Static and dynamic portfolio allocation with nonstandard utility functions

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Abstract

This article builds on the mean-variance criterion and the links with the expected utility maximization to define the optimal allocation of portfolios, and extends the results in two ways, first considers tailored made utility functions, which can be non continuous and able to capture possible preferences associated with some portfolio managers. Second, it presents results that relate to static (myopic) portfolio allocation decisions connected to dynamic settings where multi-period allocations are considered and conditions are defined to rebalance the portfolio as new information arrive. The conditions are established for the compatibility of static and dynamic decisions associated with different utility functions.

Keywords: Dynamic optimization, Expected utility maximization, Mean-variance criterion, Portfolio allocation, Step utility functions

1 Introduction

In portfolio allocation problems, the main feature is the allocation of resources to different alternatives available, in this case, different financial assets. The main objective is to choose a specific combination that is optimal by a given criterion. This problem has been studied since the research in Markowitz (1952). This author has founded what is now known as the normative portfolio theory. Due to difficulties in the implementation of the theory as a way of justifying real decisions and to a changing emphasis in the study of financial markets, there has been a continuous research on these matters. The normative approach supposes that decision agents can specify fully their utility function and know with certainty the distribution of returns.

In the normative portfolio theory and other fields, it is common to define utility functions that are analytically tractable. One of the best known is the quadratic utility function. In the portfolio theory, this utility function has an additional advantage. Decisions can be formulated in terms of the vector of means and covariance matrix associated with the returns, whatever the distribution considered. This utility function has several undesirable characteristics that led to wide ranging criticisms.

One simplification associated with the quadratic utility function is related to the equivalence between expected utility maximization and mean-variance criterion. This is not specific to this utility function. Assuming the gaussian form to the returns distribution, it can easily be demonstrated that with a negative exponential utility function there is also an equivalence between both criteria.

With other utility functions commonly used like the power and logarithmic utility functions, there is not a direct equivalence between expected utility maximization and mean-variance criteria. To overcome this problem, extensive research has been made to justify the mean-variance criteria as an approximation in this context (Levy and Markowitz, 1979; Pulley, 1981, 1983; Kallberg and Ziemba, 1983; Kroll et al, 1984; Reid and Tew, 1986). The main result is that, even when it is not the exact solution, the mean-variance criterion can give decisions that are similar to the ones obtained through these utility functions. It is in this way that most research produced recently uses the mean-variance criterion instead of the expected utility maximization paradigm. The most recent research justifying the mean-variance criterion is found in Levy and Levy (2014) and Markowitz (2014).

With a known gaussian distribution, a myopic approach can be used without much thought about the length of the period considered, differences between one and multiperiod settings and the role of portfolio reallocations. To give an example, let us designate the vector of returns by y and assume that this random vector follows a gaussian distribution with mean μ and covariance matrix Σ. The length of the period considered can easily be changed by multiplying the parameters by an appropriate factor h. More recently, several characteristics in financial markets have been highlighted that led to rethinking the straightforward rules currently proposed to define optimal portfolios.

Nowadays the distinction between short and long term decisions assumes a crucial role. Procedures defined for decision processes must be different. In a mean-variance criteria framework there is a trade-off between returns and risk, which means that the weights given to each component depend not only on the specific decision agent considered, but also on the horizon considered.

Reallocations have started to assume an important role. Reallocations are needed because there is a time-varying distribution of returns, and also, because the information available for the decision agents can change. Recent research have been produced on dynamic portfolio allocation problems, where considering several scenarios associated with the time-varying distributions for the returns, and the preferences of the investors expressed by an utility function, different rules for multi-period investments have been put forward (Birge, 2007; Skiadas, 2007; Yu et al, 2010; Bae et al, 2014; Bodnar et al, 2015b,a).

The rest of the paper is organized as follows. Section 2 presents the standard results associated with a normative approach to portfolio selection, highlighting the practical difficulties associated with its implementation and referring to some ways to overcome those difficulties, for example, with the use of robust estimators and robust optimization. Section 3 presentes a series of results that relate the mean-variance criterion with the use of non continuous utility functions. In Section 4 as the normative approach and the mean-variance criterion were developed for static settings, an extension to a dynamic setting is considered. Section 5 presents some concluding remarks.

2 The normative approach

Portfolio choice is an example of a decision taken in an environment of uncertainty. The theoretical framework specifies that rational decisions are the ones yielding from the expected utility maximization paradigm. For a more extensive treatment of portfolio choice problems; see Ingersoll (1987) and Huang and Litzenberger (1988).

The portfolio decision consists in choosing the proportions of an initial wealth to be invested in a set of available assets. With a myopic decision, supposing that the initial wealth is w_0, the decision vector is designed by $x = (x_1, \ldots, x_k)^\top$, where x_i, $i = 1, \ldots, k$, is the proportion of w_0 to be invested in the asset i, y represents the vector of random returns and the final wealth is given by $w = w_0 x^\top (1 + y)$. The aim is to choose a set of weights which to maximize the expected utility, $E(U(w))$, subject to a set of constraints.

Since Markowitz (1952) the expected utility maximization in a portfolio choice context has been replaced by the mean-variance criterion. Instead of considering different utilities for different values of the final wealth, the portfolio is chosen by weighting two conflicting characteristics to any investor, expected returns and risk. The risk is measured by the variance of returns.

The aim is to maximize expected returns for a given variance of returns, or put in another way, minimize variance of returns for a given expected return. This latest version gives rise to a quadratic programming problem. If instead of final wealth, only the vector of returns y with mean μ and covariance matrix Σ is used, the mean and variance of returns of a portfolio represented by x is given respectively by $\mu_p = x^\top \mu$ and $\sigma_p^2 = x^\top \Sigma x$. The quadratic programming problem can be formulated as

$$\underset{x}{\text{minimize}} \ \frac{1}{2} x^\top \Sigma x \qquad \text{subject to } \left\{ x^\top \mu = \mu_p^*; x^\top \mathbf{1} = 1; x \geq \mathbf{0} \right\} \tag{1}$$

where μ_p^* is the target expected return and $\mathbf{1} = (1, \ldots, 1)^\top$. By varying the target return, the efficient frontier can be obtained, which represents a set of points $\left(\sigma_p^2, \mu_p^*\right)$ associated with different optimal portfolios for different values of μ_p^*.

If $x \geq \mathbf{0}$ is not an active constraint, the optimal portfolio is given by

$$x^* = \Sigma^{-1} (\lambda_1 \mu + \lambda_2 \mathbf{1}) \tag{2}$$

where

$$\lambda_1 = \frac{\mathbf{1}^\top\Sigma^{-1}\mu}{D} - \frac{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}}{D}\mu_p^* \tag{3}$$

$$\lambda_2 = \frac{\mathbf{1}^\top\Sigma^{-1}\mu}{D}\mu_p^* - \frac{\mu^\top\Sigma^{-1}\mu}{D} \tag{4}$$

and

$$D = \mathbf{1}^\top\Sigma^{-1}\mu\mathbf{1}^\top\Sigma^{-1}\mu - \mathbf{1}^\top\Sigma^{-1}\mathbf{1}\mu^\top\Sigma^{-1}\mu \tag{5}$$

A second version of the mean-variance criterion does not use a target expected return but instead a parameter of risk aversion. The objective function includes both components that characterize this criterion, the mean and variance of returns. The quadratic programming problem can be formulated as

$$\underset{x}{\text{minimize}} \quad \frac{\lambda}{2}x^\top\Sigma x - x^\top\mu \quad \text{subject to} \quad \left\{x^\top\mathbf{1} = 1; x \geq \mathbf{0}\right\} \tag{6}$$

where $\lambda > 0$ is a parameter of risk aversion. When compared with the version presented in (1), where the efficient frontier is defined by varying μ_p^*, the same efficient frontier can be obtained by varying λ.

Sometimes, due to values assumed by the parameters, the non-negativity constraint is not active. As happens with the version presented in (1), an analytical expression for the optimal portfolio is available, which is

$$x^* = \frac{1}{\lambda}\Sigma^{-1}\left(\mu - \frac{\lambda - \mathbf{1}^\top\Sigma^{-1}\mu}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}}\mathbf{1}\right) \tag{7}$$

When the constraint $x \geq \mathbf{0}$ is active, meaning that is not possible to define portfolios using short-selling, in both cases, (1) and (6), to define optimal portfolios, numerical methods associated with quadratic programming must be used. The mean-variance criterion has close links with the expected utility maximization criterion. Two cases are presented where the exact equivalence can be established. These two cases correspond to the use of a quadratic or a negative exponential utility function.

Proposition 1 *If an investor expresses his preferences through a quadratic utility function, an equivalent criterion to the expected utility maximization criterion is the mean-variance criterion, independently of the distribution of returns considered.*

Proof. *Suppose that an investor uses the utility function given by* $U(w) = w - bw^2$, $b > 0$. *If this function is written as a second order Taylor expansion around the mean of* w, μ_w, *then*

$$U(w) = U(\mu_w) + U'(\mu_w)(w - \mu_w) + \frac{1}{2}U''(\mu_w)(w - \mu_w)^2 \tag{8}$$

Taking the expectation on both sides means that

$$E(U(w)) = U(\mu_w) + \frac{1}{2}U''(\mu_w)\sigma_w^2 \tag{9}$$

As $U''(w) = -2b < 0$ *for all* w, *then for a given target mean* μ_w, *the expected utility is maximized by choosing a portfolio which minimizes* σ_w^2. ■

Proposition 2 *When an investor expresses his preferences by a negative exponential utility function, if the vector of returns* y *follows a multivariate gaussian distribution with mean* μ *and covariance matrix* Σ, *then there is an exact equivalence between expected utility maximization and mean-variance criterion.*

Proof. *Suppose that the investor expresses his preferences through the utility function* $U(w) = -\exp(-\lambda w)$, $\lambda > 0$. *In this case, we have a version of the mean-variance criterion as it is expressed in (6). If returns follow a multivariate gaussian distribution, a given portfolio induces a gaussian distribution to the final wealth with mean* μ_w *and variance* σ_w^2. *The expected utility is given by*

$$\begin{aligned} E(U(w)) &= \int_{-\infty}^{\infty} -\exp(-\lambda w)\frac{1}{\sqrt{2\pi}\sigma_w}\exp\left(-\frac{(w-\mu_w)^2}{2\sigma_w^2}\right)dw \\ &= -\exp\left(\frac{\lambda^2}{2}\sigma_w^2 - \lambda\mu_w\right) \end{aligned} \tag{10}$$

When $E(U(w))$ *is maximized, this is equivalent to minimize* $\frac{\lambda}{2}\sigma_w^2 - \mu_w$. *The expected utility maximization criterion is equivalent to the mean-variance criterion.* ■

In the normative portfolio theory, it is assumed that investors can specify fully their utility functions and that parameters characterizing the distribution of returns are also known. In the application of the normative portfolio theory it is usually assumed that parameters can be conveniently estimated by an independent and identically distributed series of returns and the mean-variance is used instead of the expected utility maximization criterion. If the mean-variance criterion has not suffered a significant amount of criticism, the ability to deliver meaningful values to parameters used in this context has consistently been put in doubt (Kalymon, 1971; Winkler and Barry, 1975; Bawa et al, 1979; Frost and Savarino, 1986; Jorion, 1986; Frankfurter and Lamoureux, 1987; Jorion, 1991; Chopra and Ziemba, 1993; Polson and Tew, 2000; Aït-Sahalia and Brandt, 2001; Kritzman, 2006; Lee and Stefek, 2008).

There are two main factors that hamper the ability of defining parameters through simple sample estimates. The first is the role played by estimation error. The second is just the simple fact that the assumption of independent and identically distributed returns is an inappropriate one. Two ways have been considered to soften this problem, one is to consider the use of robust statistics to estimate in a more consistent way the inputs of the model through a Bayesian treatment of the parameter uncertainty (Bawa et al, 1979; Frost and Savarino, 1986; Jorion, 1986; Frankfurter and Lamoureux, 1987; Jorion, 1991; Polson and Tew, 2000; Kan and Zhou, 2007; DeMiguel and Nogales, 2009), the other is addressing the problem assuming a pessimist approach related to the optimization problem subjacent to the decision process, and adopting robust optimization approaches (Goldfarb and Iyengar, 2003; Tütüncü and Koenig, 2004; Fabozzi et al, 2007a,b; Schöttle, 2007; Ben-Tal et al, 2010; Schöttle et al, 2010; Gregory et al, 2011; Chen and Kwon, 2012; Ye et al, 2012; Scutellà and Recchia, 2013; Kolm et al, 2014; Fliege and Werner, 2014).

3 Tailored utility functions

The utility functions mentioned above are commonly used in financial literature. They fulfil basic requirements when associated with investors that prefer more to less and are risk averse. Mathematically this can be expressed using first and second derivatives of the utility function, $U'(w) > 0$ and $U''(w) < 0$. The first derivative being always positive ensures that more is preferred to less. The second derivative being always negative implies that investors are risk averse.

An important role played by utility functions referred above is their ability to characterize the risk aversion associated with each investor. Two important measures of risk have been used in this context, Absolute Risk Aversion (ARA) and Relative Risk Aversion (RRA) indices. They are defined for each continuous utility function considered as $ARA(w) = -U''(w)/U'(w)$ and $RRA(w) = -wU''(w)/U'(w)$. These indices have been used to highlight some difficulties associated with the use of those utility functions. It is commonly accepted that is reasonable to expect that general investors can be characterized by a constant absolute risk aversion and a decreasing relative risk aversion as a function of the level of wealth. The quadratic utility function possesses the undesirable characteristic that the relative risk aversion is increasing with the wealth. With the negative exponential utility function, absolute and relative risk aversion levels are constant, equal to λ. Only with the logarithmic utility function the relative risk aversion level is decreasing with the wealth. However, with a logarithmic utility function it may be difficult to model differences in risk aversion for different investors with the same level of wealth. This can be achieved by a negative exponential utility function, which also has a direct link with the mean-variance criterion.

It was demonstrated above that with a negative utility function, there is a direct equivalence between expected utility maximization and mean-variance criterion. In this, with an objective function expressed as $x^\top \mu - \frac{\lambda}{2} x^\top \Sigma x$, the portfolio considered depends not only on parameters characterizing the distribution of returns but also on the parameter of risk aversion λ. In this context, it may be difficult to define a parameter of risk aversion for each investor. What has been proposed is to obtain a set of optimal portfolios for different values of λ, the efficient frontier. Investors choose a portfolio not by specifying directly their risk aversion parameter but instead by choosing appropriate means and variances available through the definition of the efficient frontier. To some investors this can be as much difficult as the specification of a parameter of risk aversion.

We present in this section a different class of utility functions that can be linked with the mean-variance criterion and only needs the specification of a parameter that should be intuitive to each investor, a target return. We think that this utility function can be useful to specific decision makers, namely, portfolio managers.

We specify a step utility function with an associated target return b. With so much noise associated with the data gathered from financial markets, it is difficult to specify meaningful utilities for a possible wide range of values that the variables of interest can assume. With a portfolio manager whose performance is evaluated at most at some discrete criteria, it is difficult to argue that he manages a given portfolio through a continuous utility function.

The form of the utility function presented here is given by

$$U(y,x,b) = \begin{cases} 1, \ x^\top y \geq b \\ 0, \ x^\top y < b \end{cases} \tag{11}$$

This utility function defines two patterns of utility, one associated with portfolio returns, $x^\top y$, that are greater than the target return, b, the other to the ones that are lower. In this utility function, instead of specifying a parameter that defines directly the absolute or relative risk aversion, the decision agent only needs to specify the target return.

As in previous cases, optimal portfolios are defined through the expected utility maximization. Optimal portfolios are chosen by

$$\underset{x}{\text{maximize}} \ E(U(y,x,b)) \quad \text{subject to} \ \left\{x^\top \mathbf{1} = 1; x \geq \mathbf{0}\right\} \tag{12}$$

Proposition 3 *Supposing that the utility structure for a given decision agent in financial markets is expressed by the function in (11), assuming that the vector of returns follows a multivariate gaussian distribution with mean μ and covariance Σ, there is a version of the mean-variance criterion that is equivalent to the expected utility maximization criterion.*

Proof. *Without taking into account the constraints, defining $F(\cdot)$ the distribution function of portfolio returns with mean $\mu_p = x^\top \mu$ and variance $\sigma_p^2 = x^\top \Sigma x$, the expected utility in (12) is given by $E(U(y,x,b)) = 1 - F(b)$. As $F(\cdot)$ is fully defined by $\mu_p = x^\top \mu$ and $\sigma_p^2 = x^\top \Sigma x$, the expected utility can be obtained as a function of these parameters and b, then this represents a specific version of the mean-variance criterion.* ■

Example 4 *To illustrate the results presented in the proposition 3, let us suppose that y is a vector containing two independent gaussian random variables with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2. If the aim is to use the mean-variance criterion by maximizing $x^\top \mu - \frac{\lambda}{2} x^\top \Sigma x$ for a given parameter of risk, λ, assuming that the optimal portfolio is represented by $x^* = (x_1^*, x_2^*)^\top$ with $x_2^* = 1 - x_1^*$, the solution can be expressed as*

$$x_1^* = \frac{\lambda \sigma_2^2 + (\mu_1 - \mu_2)}{\lambda \left(\sigma_1^2 + \sigma_2^2\right)} \tag{13}$$

On the other hand, when the utility function in (11) is used instead, with the remaining assumptions, the optimal portfolio is defined in a way that

$$x_1^* = \frac{(b-\mu_1)\,\sigma_2^2}{(b-\mu_1)\,\sigma_2^2 + (b-\mu_2)\,\sigma_1^2} \tag{14}$$

In example 4, in both mean-variance criterion and expected utility maximization through the step utility function, optimal portfolios can be obtained as a function of the parameters characterizing the distribution of returns plus an additional parameter. In the original mean-variance criterion this parameter represents a measure of risk aversion, which in some circumstances can be difficult to define. With a step utility function, the additional parameter corresponds to the target return, much easier to define.

A portfolio that has received some attention recently is the minimum variance portfolio. This is obtained by minimizing $\frac{1}{2}x^\top \Sigma x$ subject to $x^\top \mathbf{1} = 1$. The solution x^* of this problem is given by

$$x^* = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}} \tag{15}$$

This formulation derives from assuming that $\mu = \mathbf{0}$ in the usual mean-variance criterion. Less obvious is the result that a minimum variance portfolio is also obtained by considering $\mu_1 = \cdots = \mu_k$. There is no possibility of exploring high levels of returns. The optimal strategy is just to minimize the risk measured by the variance of portfolio returns. These features are present in both models using the original mean-variance criterion or the expected utility maximization criterion through a step utility function. Although in certain aspects, both criteria are similar when a gaussian distribution to returns is considered, the step utility function has some intuitive appeal.

Example 5 (Continued from example 4) *An important aspect of portfolio construction is the possibility of diversification as a way of reducing the risks involved. An important role is played by the correlation between different returns. Assuming as in Example 4 a gaussian distribution, let y_1 and y_2 be correlated with an associated covariance σ_{12}. In the mean-variance criterion and step utility function, optimal portfolios imply that*

$$x_1^* = \frac{\lambda\left(\sigma_2^2 - \sigma_{12}\right) + (\mu_1 - \mu_2)}{\lambda\left(\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}\right)} \tag{16}$$

and

$$x_1^* = \frac{(b-\mu_1)\,\sigma_2^2 + (\mu_2 - b)\,\sigma_{12}}{(b-\mu_2)\,\sigma_1^2 + (b-\mu_1)\,\sigma_2^2 + (\mu_1+\mu_2-2b)\,\sigma_{12}} \tag{17}$$

respectively.

In this example, the role played by covariances can be taken into account, but most of the relevant features are just a straightforward extension of the example given previously. It can be argued that the step utility function in (11) gives essentially the same result as the mean-variance criterion expressed in (1). In both, there is a target return that is included into the analysis. However, we must notice that there is a clear difference between both formulations. In the mean-variance criterion, the target return is included in a constraint and when the problem is feasible, the expected return for the optimal portfolio is equal to the target expected return. It is clear that the target expected return must be fixed between the minimum and maximum expected returns considered individually.

When the step utility function is considered, the target return is a component defining the utility function, which means that the target return can be viewed in probabilistic terms. An optimal portfolio intends to maximize the probability of obtaining returns higher than the target return. In this sense, there is no constraint in the definition of the target return. What can happen is that sometimes it is possible to find a portfolio with a high probability of obtaining returns higher than the target return. Other times this is not possible. However, in both cases, it is possible to define an optimal portfolio. These facts can easily be appreciated through in the following example.

Example 6 *As in example 4, assuming that y is a vector containing two independent gaussian random variables with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2. If a portfolio is chosen using the criterion expressed in (1), ignoring the constraint $x \geq \mathbf{0}$ and $\mu_p^* = b$, the same target expected return as expressed for a step utility function, the optimal portfolio $x^* = (x_1^*, x_2^*)^\top$, where $x_1^* = 1 - x_2^*$, can now be defined as*

$$x_1^* = \frac{b-\mu_2}{\mu_1-\mu_2} \tag{18}$$

The optimal portfolio defined here can be compared with the one defined in (14), which are clearly different. Another aspect must be noticed in this simple example, in contrast to the one expressed in the example 4, variances play no role.

A straightforward extension of the step utility presented in (11) is the one that considers three instead of two patterns of utility. The objective of this extension is to allow an asset that is available to all investors, with a very particular characteristic of being riskless, to be included into the analysis. A third pattern of utility is defined through the parameter α, where $0 < \alpha < 1$. If the return associated with a riskless return is designed by r_f, the utility function proposed here is defined as

$$U(x,y,\alpha,b) = \begin{cases} 1, & x^\top y \geq b \\ \alpha, & r_f \leq x^\top y < b \\ 0, & x^\top y < r_f \end{cases} \tag{19}$$

As this utility function is just an extension of the one expressed in (11), and also defining $F(\cdot)$ as the distribution function associated with portfolio returns, the expected utility can be expressed in general terms by

$$E(U(x,y,\alpha,b)) = 1 + (\alpha - 1) F(b) - \alpha F(r_f) \tag{20}$$

Example 7 (Continued from example 4) *Using the same assumptions as in example 4 and also expressions (19) and (20), it is straightforward to show that an optimal portfolio can be defined by*

$$x_1^* = \frac{\mu_1 \sigma_2^2 - \mu_2 (1 - 2\alpha) \sigma_{12} - bB(\alpha - 1) - \left(Br_f + 2\mu_1 \sigma_2^2\right) \alpha}{(\mu_2 A + \mu_1 B + bC)(\alpha - 1) + \left(\mu_2 A + \mu_1 B + r_f C\right) \alpha} \tag{21}$$

where $A = \sigma_{12} - \sigma_1^2$, $B = \sigma_{12} - \sigma_2^2$ *and* $C = \sigma_1^2 - 2\sigma_{12} + \sigma_2^2$.

This last example illustrates the potential richness associated with this class of utility functions, which associated with a gaussian distribution can produce a larger variety of behaviours when compared with the standard mean-variance criterion.

A straightforward extension of the results presented in this section is when Student$-t$ instead of gaussian distributions are considered to the returns. As with the gaussian distribution, the expected utilities associated with (11) and (19) can be expressed as $E(U(y,x,b)) = 1 - F(b)$ and $E(U(x,y,\alpha,b)) = 1 + (\alpha - 1) F(b) - \alpha F(r_f)$ respectively. In this case, it is also straightforward to obtain the optimal portfolios. However, in contrast to the case where the returns follow a gaussian distribution, no analytical results can be obtained and numerical methods must be used.

4 Dynamic portfolio allocation

In this section we make some tentative steps towards the definition of dynamic portfolio allocation rules where the financial risk component assumes a predominant role. Inserted in a Bayesian framework, dynamic portfolio allocation has been analysed in Aguilar and West (2000) and Polson and Tew (2000). This problem can be inserted in a multi-period problem.

When the horizon established is at n, the aim is to define a series of functions $\{x_1,\ldots,x_{n-1}\}$ that form an optimal path with the aim of maximizing the expected utility associated with the wealth at n. The idea is that information can be gathered and the optimal path depend on this information. In Aguilar and West (2000) the sequential approach is only dependent on the statistical procedure used to estimate Σ_t. In a Bayesian approach, using a quadratic loss function, the estimate of Σ_t is obtained by the mean of the posterior distribution of $\Sigma_t|D_t$, where $D_t = \{y_1,\ldots,y_t\}$ represents all the information until t. These authors have applied just a simple set of sequential myopic portfolio allocation rules. Polson and Tew (2000) applied a sequential portfolio framework but recognized the important role played by portfolio constraints and estimation errors.

A common constraint associated with reallocations is the presence of transaction costs. The authors, even using daily returns to take into account characteristics like fat tails and non-stationary distributions, in their simulations applied reallocations only in biannual basis, recognizing that shorter periods might be counterproductive due to the presence of transaction costs. Due to estimation errors, it was recognized that the weights in portfolios can assume extreme values, leading to less well diversified portfolios. This can be overcome by imposing boundary constraints associated with those weights.

The application of static type of rules in a dynamic context when related with portfolio allocation procedures can be an interesting statistical exercise, but, needs some refinement when the aim is to define appropriate sequential portfolio allocation rules. More recently new research has been produced associated with dynamic portfolio allocation (Skiadas, 2007; Yu et al, 2010; Bodnar et al, 2015a,b). Dynamic portfolio allocation is substantially more harder to treat in comparison with the static case, where a decision vector is defined. In a dynamic setting a set of decision rules must be established. It is much more difficult to establish the conditions for the returns and for the utility functions as a way of defining closed-forms for such set of rules. For example, Bodnar et al (2015a) found such rules but assuming a quadratic utility function, which has some disadvantages as aforementioned. Also, Bodnar et al (2015b) developed a set of results, but assuming the predictability of the returns, which is also a very strong assumption. We build on these, presenting some results associated with the dynamic portfolio allocation and the relation of the mean-variance criterion with the successive application of an one-period decision rule.

As was referred in the previous section, where a different class of utility functions was proposed and an example was given related to a possible decision agent willing to use these utility functions, we argue that the rules defined to the static case need to be rethought when applied to a dynamic case, namely, in short term settings.

Dynamic portfolio allocation allows the use of dynamic programming as well as backward induction techniques. For a more detailed exposition of these techniques, some of them applied to portfolio allocation problems; see DeGroot (1970), Bertsekas (1976), Whittle (1986) and Cyert and DeGroot (1987). These techniques are used to define decisions in a multi-period setting where between the initial and end period, the redefinition of a decision taken previously can be performed. With dynamic portfolio allocation, when the aim is to maximize the expected utility at the end of period n and restrictions related short-sales and transaction costs are not considered, for a certain class of utility functions with very restrictive conditions, it can be demonstrated that the optimal path of decisions coincides with a sequential set of myopic decisions. At each $t-1$ a prediction related to Σ_t is available and the portfolio is chosen as the aim was to maximize only the expected utility at t. When more realistic scenarios are considered with short-sales constraints as well as transaction costs, a sequential set of myopic decisions may not be the optimal strategy.

Dynamic portfolio allocation is essentially a mult-iperiod decision problem. This can be treated as an optimization problem under uncertainty, which has the special characteristics related to the need to take into account the role played by the risk and the possibility of information gathering during the decision process. There are circumstances where a sequential set of decisions in a multi-period decision context is equivalent to a sequence of myopic decisions. However, as we will demonstrate below, this needs a specific formulation and small deviations will yield a different optimal path when compared with a sequence of myopic decisions.

To demonstrate the differences between truly sequential procedures and a sequence of single-period decisions, we resort to concepts associated with dynamic programming and backward induction. One of the most important concept in this context is known as the Bellman's Principle of Optimality, which can be expressed as:

Definition 8 *An optimal sequential decision policy to be evaluated at the period n, supposing that a prior $f(\theta)$ was specified at 0, and with $m < n$, already m observations $y_1, \ldots, y_m$ were obtained and the posterior distribution to $\theta | y_1, \ldots, y_m$ was defined, then independently from the decisions already taken, the continuation of the optimal policy must be the optimal sequential policy for the problem beginning with $f(\theta | y_1, \ldots, y_m)$ as a prior for the remaining $n - m$ stages.*

Another important result is given in the following theorem which establishes the recursive nature of an optimization process over time. The decision at time t is denoted by x_t and the partial sequence of decisions $x_1, \ldots, x_t$ by χ_t. When an horizon n is considered, the cost function can be considered as a function of the decisions made over this period, $C(x_1, \ldots, x_{n-1}) = C(\chi_{n-1})$.

Theorem 9 (Whittle 1986) *Define the functions*

$$C(\chi_{n-1}, t) = \inf_{x_t, \ldots, x_{n-1}} C(\chi_{n-1}) \tag{22}$$

Then this obey the recursion

$$C(\chi_{n-1}, t) = \inf_{x_t} C(\chi_{n-1}, t+1) \qquad (0 < t < n) \tag{23}$$

with terminal evaluation

$$C(\chi_{n-1}, n) = C(\chi_{n-1}) \tag{24}$$

Furthermore, the minimizing value of x_t in (23) is the optimal value of x_t for prescribed χ_{n-1}.

We presented theorem 9 with a general formulation where $C(\cdot)$ is denoted as a cost function, which can be generalized to utility or expected utility functions and instead of $\inf C(\cdot)$ we have $\sup U(\cdot)$ or $\sup E(U(\cdot))$.

The results presented above allow the definition of an important concept which will be useful in an sequential portfolio optimization framework. This concept is known by backward induction. Supposing that the established horizon is n, and a single-period type of analysis can be defined at $n-1$, moving backwards one period, the decision can be defined at $n-2$ assuming that the optimal decision was also defined at $t-1$. This process can in principle be iterated until the period 1, defining in this way a sequence of optimal decisions $x_1^*, \ldots, x_{n-1}^*$.

In a portfolio allocation context, defining a target horizon n, the sequence of decisions, $x_1^*, \ldots, x_{n-1}^*$, must be chosen as a way of maximizing the expected utility at n, $E\left(U\left(w_n\right)\right)$. The procedures presented above associated with the recursive nature of optimization over time can be redefined in is context as

$$\begin{aligned} \phi_{n-1}\left(x_{n-1}, w_{n-1}\right) &= \underset{x_{n-1}}{\operatorname{maximize}}\, E\left(U\left(w_n\right)\right) & (25) \\ \phi_t\left(x_t, w_t\right) &= \underset{x_t}{\operatorname{maximize}}\, E\left(\phi_{t+1}\left(x_{t+1}, w_{t+1}\right)\right) & (26) \end{aligned}$$

By starting with a single-period decision at $n-1$, and then moving backwards, a set of rules can be defined which represent the optimal sequence of decisions to be taken in a multi-period portfolio allocation problem.

In the statistical literature, or even in the financial literature, when portfolio allocation problems are considered, rarely the expected utility maximization paradigm is used directly and the mean-variance criterion is used instead. It was already referred the setting needed to obtain an exact equivalence between portfolios in the expected utility maximization and mean-variance criterion.

Proposition 10 *Suppose that an investor expresses his preferences through a negative exponential utility function, $U\left(w_t\right) = -\exp\left(-\lambda w_t\right)$, with $\lambda > 0$. If the investor has the possibility of reallocating at each t the portfolio defined at $t-1$ and the portfolio is evaluated at n, the optimal sequential path corresponds to a sequence of single-period decision rules if and only if a riskless asset is available and the returns of the risky assets at each t are independent.*

Proof. *Let us consider a two-period setting and suppose that there are only two assets available at each t, a risky asset with return y_t and a riskless asset with return r_f. The decision at t is denoted by u_t , which is the amount to invest in the risky asset. The amount to invest in the riskless asset is given by $w_t - u_t$. In this, the wealth at t is given by*

$$\begin{aligned} w_t &= \left(w_{t-1} - u_{t-1}\right) r_f + u_{t-1} y_t \\ &= w_{t-1} r_f + u_{t-1}\left(y_t - r_f\right) & (27) \end{aligned}$$

Using the two-period setting, the definition of sequential set of decision functions start by considering

$$\underset{u_1}{\text{maximize}}\, E\left(U\left(w_2\right)\right) \tag{28}$$

and then

$$\underset{u_0}{\text{maximize}}\, E\left(\underset{u_1^*}{\text{maximize}}\, E\left(U\left(w_2\right)\right)\right) \tag{29}$$

With a negative exponential utility function, in (28), $E\left(U\left(w_2\right)\right)$ *can be expressed as*

$$E\left(U\left(w_2\right)\right) = E\left(-\exp\left(-\lambda w_1\right)\right) E\left(-\exp\left(-\lambda u_1\left(y_2 - r_f\right)\right)\right) \tag{30}$$

The value of u_1^* *is not a function of* w_1 *and in (29), assuming that the optimal decision is taken at* 1, u_0^* *is chosen as only if the maximization of* $E\left(-\exp\left(-\lambda w_1\right)\right)$ *is considered, which is equivalent to consider two separate single-period decisions. If the returns of risky assets were not independent, in this case* y_1 *is not independent of* y_2*, and knowing that* $w_1 = w_0 r + u_0\left(y_1 - r_f\right)$*, then the factorization expressed in (30) would not apply, which means that* u_0 *and* u_1 *should be defined jointly. When a riskless asset is not considered, using a portfolio with two risky assets that at t the respective returns are denoted by* y_{1t} *and* y_{2t} *and the associated with the asset* 2 *is denoted by* u_{2t}*, in a two-period setting*

$$w_2 = w_1 y_{12} + u_{22}\left(y_{22} - y_{12}\right) \tag{31}$$

as y_{12} *is not independent from* $\left(y_{22} - y_{12}\right)$*, it means that here the factorization used in (30) cannot be applied.* ■

Corollary 11 *Using the conditions expressed in the proposition 10, the same type of results as expressed in this proposition apply to the case of a logarithmic utility function* $U\left(w_t\right) = \ln\left(w_t\right)$.

Proof. *This can easily be demonstrated modifying slightly the decision variables which are now the proportions of an initial wealth,* w_t*, to be invested in an asset* x_t*. In this the wealth at* 2 *is given by* $w_2 = w_1\left(r_f + x_1\left(y_2 - r_f\right)\right)$ *and* $E\left(\ln\left(w_2\right)\right) = E\left(\ln\left(w_1\right)\right) + E\left(\ln\left(r_f + x_1\left(y_2 - r_f\right)\right)\right)$*, which imply that the results presented in proposition 10 also apply here.* ■

If the mean-variance criterion is used as the equivalent decision with a negative exponential function, the procedures implemented Aguilar and West (2000) and Polson and Tew (2000) did not use a truly sequential approach. There are two conditions that do not apply, lack of a riskless asset and lack of independence between successive returns, which means that the procedures applied were just a sequence of single-period decisions rules and not a multi-period decision, which means that comparisons made using the wealth at the end of the simulation process cannot interpreted properly, and does not direct us to the best portfolio allocation.

Usually the link between statistical models used to define the evolution of a mean or variance processes and the models defined to portfolio allocation problems is made through a quadratic loss function. When the posterior distributions to the parameters of interest are defined, using a quadratic loss function, the estimates chosen are the means of the respective distributions.

This strategy has been implemented when long horizons but also short horizons are considered in portfolio allocation problems. When there are changes in the means of the returns the reallocations allow higher returns to be obtained. However, when only the variances are considered, reallocations are needed to adjust the portfolios to the levels of risk desired, but they have not a major influence on the expected returns of the portfolio. By comparing the forecasting ability related to the variance of returns through the final wealth obtained to the portfolio seems to be a spurious comparison.

When only the volatility evolution is considered with the aim of controlling the financial risk, other loss functions like a step utility function may be relevant in this context of linking statistical models with financial decisions. These will allow that only significant increases in the volatility to be considered, which means that reallocations need not to be so frequent. This possibly can alleviate the difficult task of establishing the link between a sequence of single-period portfolio decisions and a multi-period decision.

5 Conclusions

The mean-variance criterion is widespread in many situation when portfolio allocation decisions are considered. Even subject to criticisms, the main issue is not the validity of the decisions proposed, but instead the oversimplified form that the criterion is applied in practice. Ignoring the uncertainty associated with the parameters, investors' risk aversion and the time-varying distribution of the returns, the lack of distinction between one- and multi-period decisions can lead to meaningless results. However, for a given set of true values for the parameters, by defining the portfolios efficient frontier, for a specific investor, the optimal portfolio must be represented by a point in such frontier.

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