Macroeconomic effects of loss aversion in a signal extraction model[1]

Giuseppe Ciccarone[2] Enrico Marchetti[3]

Abstract

We add some elements of prospect theory to an analytically tractable version of Lucas's "islands" model and show that the inclusion of reference dependence, declining sensitivity and loss aversion into the agents' utility function leads to three main results. First, the equilibrium labor supply and the natural level of output are negatively affected by the presence of behavioral elements, whereas the cyclical response of output to a monetary shock remains unaltered. Second, the expected utility of a representative agent is generally lower than that obtained when loss aversion is absent. Third, the presence of loss aversion eliminates the paradoxical increase in expected utility that may be generated, in the standard model, by an increase in monetary policy uncertainty.

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[2]Corresponding author; Department of Economics and Law, Sapienza University of Rome, Via del Castro Laurenziano, 9 00161 Rome, Italy. E-mail: giuseppe.ciccarone@uniroma1.it.

[3]University of Naples Partehnope, Naples, Italy.

1 Introduction

The aim of this paper is to understand the effects produced by the introduction of behavioral elements into microfuonded macroeconomic models. Even though the wide corpus of empirical evidence supporting prospect theory (PT) has facilitated its successful application to a variety of economic problems and it today stands as one of the most promising frontiers of economic research, the attempt to understand its macroeconomic implications is still in its infancy.[4] In order to introduce the argument in the simplest possible way, we modify the well-kown Lucas's "islands" model (1972; 1973) so as to take into account some of the main elements of PT in an economic context where only information is imperfect and only (unexpected) monetary policy may affect the level of output.

Our investigation is based on several motivations. First, a unanimous belief in modern macroeconomics is that the level of potential output depends only on the degree of market imperfections which are present in the economy. Friedman's (1968, p. 8, italics added) well known definition of the natural rate of unemployment, associated to the natural level of output, corresponds to "the level that would be ground out by the Walrasian system of general equilibrium equations, provided there is imbedded in them the actual structural characteristics of the labor and commodity markets, including *market imperfections*, stochastic variability in demands and supplies, the cost of gathering information about job vacancies and labor availabilities, the costs of mobility, and so on". Friedman (1968, p. 9) also believed that "many of the market characteristics that determine its level are man-made and policy-made". Even though the notion of potential output has experienced several changes of emphasis in subsequent literature – with the associated flourishing of acronyms such as NAIRU (Not Accelerating Inflation Rate of Unemployment) or NIRU (Not Inflationary Rate of Unemployment) – this fundamental view has never been challenged. On the contrary, it has been placed at the foundations of the most recent RBC and New Keynesian DSGE models in the explanation of the business cycle and in the definition of optimal policy. In the latter context, the existing degree of monopoly in the good markets and other market frictions are the only determinants of potential output. We aim at showing that this is true only if agents conform to the tenets of standard expected utility theory. By dropping this assumption, we formally demonstrate that potential output may also be affected by behavioral elements in a framework which allows us to uniquely determine equilibrium values for the relevant macroeconomic variables.

Second, in the light of the wide evidence that financial decisions are influenced by several cognitive distortions (e.g., Cuthbertson et al. 2007; Barberis et al. 2001), the inclusion of PT in monetary models should be reckoned as a necessary step in the attempt to add realism and depth to monetary policy analysis. For example, Camerer and Loewenstein (2003, p. 39) suggest that the Lucas's islands model "can be interpreted as implicitly behavioral" as the islands can be seen as a metaphor for the limits of agents' minds. This supports our

[4] For example, Shafir, Diamond and Tversky (1997) and Fher and Tyran (2001) investigate (mainly experimentally) the ability of PT to explain money illusion and its macroeconomic consequences, price stickiness in particular. In a recent study, Gaffeo et al (2010) insert loss aversion into a New Keynesian DSGE model; they aim at providing an explanation of the asymmetric reaction of output and prices to monetary policy innovations over contractions and expansions in the business cycle.

adoption of this model. Furthermore, the inclusion of PT into the Lucas's framework should also affect, in a relevant way, the results obtained by the vast literature on optimal-strategic monetary policy which employs a "surprise inflation" aggregate supply curve.[5] Finally, the welfare effects of monetary policy could well be influenced by behavioral elements.

More in details, we introduce into Bénassy's (1999) analytically tractable version of the Lucas's (1972) signal extraction model the assumptions of: **i)** *reference dependence*, i.e., the carriers of utility are wealth gains and losses relative to some reference point; **ii)** *declining sensitivity*, i.e., the utility function is concave in the domain of gains and convex in the domain of losses; **iii)** *loss aversion*, i.e., losses are more salient than gains.[6] Within this context, we obtain three main results. First, the change we introduce in the agents' utility functions affects their equilibrium labor supply[7] and the natural level of output, but not the cyclical response of output to a monetary shock. In particular, a computational analysis allows us to show that, for a wide range of parameters' values consistent with the empirical evidence, potential output in our model is lower than that which results in the original one. Second, the expected utility of a representative agent inhabiting our model economy is generally lower than that obtained when loss aversion is absent. Third, the presence of loss aversion eliminates the paradoxical improvement in expected utility that may be generated by an *increase* in monetary policy uncertainty (Polemarchakis and Weiss 1977; Bénassy 1999).

The paper is structured as follows. In the next section we describe the model and compute equilibrium values of labor and consumption in a representative island. In section 3 we present the macroeconomic equilibrium of the economy and discuss the effects of loss aversion on the aggregate level of output. In section 4 we compute the expected utility of the representative agent, together with the effects of monetary policy uncertainty on this measure of welfare. Section 5 illustrates our computational analysis and section 6 concludes.

2 Loss aversion in a structural signal extraction model

The economy is made up of J isolated subsectors (islands) and each island j operates in a decentralized manner. Islands are hit by various shocks, some of which are correlated across islands. The representative island j, is an overlapping generations economy, where the representative agent living at time t has the biperiodal (expected) utility function:

$$E_t\left(U_{j,t}\right) = E_t\left(\frac{C_{j,t+1}^{\alpha}}{\alpha}\right) - \frac{L_{j,t}^{\psi}}{\psi}$$

[5] Stemming form the calssical works of Barro and Gordon (1983) and Rogoff (1985), among many others.

[6] This is in line with various applications of PT that consider these three elements. The complete version of PT (or, more properly, *cumulative* prospect theory - Tversky and Kahneman 1992) also includes *non-linear weighting of probabilities* (i.e., agents facing uncertain situations overweight small probabilities but underweight large ones) and *susceptibility to framing effects* (i.e., agents' preferences are influenced by the way a problem is presented). For an exhaustive discussion see, e.g., Wakker (2010).

[7] This result highlights an impact of PT on the labor supply which is different from those discussed in previous studies, such as Camerer (2000), Camerer et al. (1998) and Oettinger (1999), focusing on the short term elasticity of the labor supply.

where C is consumption (taking place when the agent is "old"), L is the labor supply (offered when the agent is "young"), $\psi > 1$ and $1 \geq \alpha > 0$ are parameters.[8] The agent is the owner of the island's firm and, at time t, produces output $Y_{j,t}$ (which is sold at price $P_{j,t}$) according to the production function:

$$Y_{j,t} = L_{j,t}$$

The economy's population is described by an i.i.d. log-normally distributed stochastic variable $N_{j,t}$, where $n_{j,t} = \ln N_{j,t} \sim N\left(0, \sigma_n^2\right)$.

The agent can transfer wealth produced when young to the next period by accumulating money, which is issued by an external and economy-wide public authority. This authority can alter the money stock period by period in every island. However, immediately after the monetary shock the population of old agents redistributes among the islands so that the new nominal money stock is the same. This implies that the monetary shock produces the same effect in every island. The money supply follows the rule:

$$M_{j,t} = X_t M_{j,t-1}$$

where $M_{j,t}$ is the amount of money held by the island's old generation at time t and X_t is the exogenous (common) monetary shock, which is assumed to be unknown to the private agents and log-normally distributed, with $x_t = \ln X_t \sim N\left(0, \sigma_x^2\right)$ in all the islands.[9].

The economy is thus hit by two different type of shock: a 'real' one ($N_{j,t}$) and a 'nominal' one (X_t), and the agents on each island are unable to properly understand the nature of the shock occurring at time t. This generates the well-known signal extraction problem.

The biperiodal budget constraint of the representative agent is given by:

$$C_{j,t+1}P_{j,t+1} = X_{t+1}P_{j,t}L_{j,t} = \lambda_{j,t}X_{t+1} \tag{1}$$

where $\lambda_{j,t}$ is the agent's demand for money to be carried over to next period. This amount of money is acquired by selling production ($P_{j,t}L_{j,t}$) to the old generation.

Given perfect competition in each market, the equilibrium condition on island j is:

$$X_t M_{j,t-1} = N_{j,t}P_{j,t}Y_{j,t} = N_{j,t}P_{j,t}L_{j,t}$$

This equation says that the purchase made by the old agents at date t, $X_t M_{j,t-1}$, must be equal to the nominal value of production sold by the young agents at the same date: $N_{j,t}P_{j,t}Y_{j,t}$, where $N_{j,t}$ is the population of island j.

In this framework, the role of money as the only storage of value allows us to introduce the basic elements of PT by adopting Barberis et al.'s (2001, p.2) idea that the agent "derives direct utility not only from consumption levels, but also from changes in the value

[8] We adhere to the common assumption that consumption and leisure are gross substitutes, which corresponds to $\alpha > 0$. This is also a required assumption in versions of PT adopting the power utility specification. In this case utility must be defined at zero, i.e., when there are no no gains or losses; see equation (3).

[9] It is important to note that σ_x^2 is a policy parameter that can be chosen by the authority.

of his financial wealth". More precisely, disregarding for simplicity the index j, we assume that the expected utility function of the representative agent is:

$$E_t\left(U_t\right) = E_t\left(\frac{C_{t+1}^{\alpha}}{\alpha}\right) - \frac{L_t^{\psi}}{\psi} + \beta E_t\left[\frac{1}{\alpha}v\left(R_{t+1}, R_{\mathrm{ref}}\right)\right] \tag{2}$$

This equation states that at time t the agent cares not only about his expected consumption level C_{t+1} *per se*, but also about his expected real wealth R_{t+1}, as compared to a reference point R_{ref}. As in Barberis et al. (2001), the parameter $\beta \geq 0$ measures the importance of gains and losses in the utility function relative to that of consumption *per se* (see the discussion below). The agent's real wealth at time $t+1$ is given by its real money holdings, λ_t/P_{t+1}, multiplied by the monetary shock X_{t+1}; this product corresponds to the level of real wealth that can be used to purchase consumption when old. Assuming monetary equilibrium at the island level we hence write:

$$R_{t+1} = \frac{X_{t+1}\lambda_t}{P_{t+1}}$$

so that X_{t+1} can be interpreted as a stochastic gross rate of return on real wealth. We take as the reference point R_{ref} the amount of real asset (money) that would be obtained if no monetary shock occurred, i.e., $X_{t+1} = 1$, or $M_{t+1} = M_t$, so that:

$$R_{\mathrm{ref}} = \frac{\lambda_t}{P_{t+1}}$$

We define the PT component of the agent's utility function in the following way:

$$v\left(X_{t+1}, \lambda_t\right) = \begin{cases} \left(\frac{\lambda_t X_{t+1}}{P_{t+1}} - \frac{\lambda_t}{P_{t+1}}\right)^{\alpha} & \text{for } \frac{\lambda_t X_{t+1}}{P_{t+1}} - \frac{\lambda_t}{P_{t+1}} \geq 0 \\ -\theta\left[-\left(\frac{\lambda_t X_{t+1}}{P_{t+1}} - \frac{\lambda_t}{P_{t+1}}\right)\right]^{\alpha} & \text{for } \frac{\lambda_t X_{t+1}}{P_{t+1}} - \frac{\lambda_t}{P_{t+1}} < 0 \end{cases} \tag{3}$$

We are hence assuming that the behavioral component, v, and the "standard" component, C_{t+1}^{α}/α, are basically the same function over consumption (given the agent's budget constraint and the overlapping generations structure of the model), with the difference that the argument in v is a gain/loss with respect to some reference level.

As it is usually assumed in PT, the coefficient $\theta > 1$ represents the effect of loss aversion. In general, it would be reasonable to assume that the agent's utility is affected not only by the level of gains or losses in $t+1$, but also by his prior investment performance. Nevertheless (as suggested by Barberis et al. 2001), the hypothesis expressed in (3), which we consider as a first step towards a more general analysis, is the simplest way to include a behavioral element into the utility function of the agent. It is also coherent with the idea that, in this overlapping generation framework, the representative agent in an island does not have a prior history of gains or losses, as he faces an "investment" decision spanning only two consecutive periods. In this sense the agent's utility of a gain or loss does not depend on his previous investment performance.

Another relevant assumption is that the curvature coefficient α in (3) is the same for the two components C_{t+1}^{α}/α and v. Even though it would be more general to envisage different parameters for these two components, two arguments support our choice. First, if this were the case, the model would possess an economically meaningful solution, but it could not be solved in closed form. Second, as both components refer to expected consumption (with the term β allowing us to better differentiate between the two), it is reasonable to assume that they affect utility according to the same functional form.[10]

From the budget constraint (equation (1)) we obtain $\lambda_t = L_t P_t$. Substituting into equation (3) we have:

$$L_t^\alpha \cdot \hat{v} = L_t^\alpha \cdot \left(\frac{P_t}{P_{t+1}}\right)^\alpha \begin{cases} (X_{t+1}-1)^\alpha & \text{for } X_{t+1} \geq 1 \\ -\theta\left[-(X_{t+1}-1)\right]^\alpha & \text{for } X_{t+1} < 1 \end{cases}$$

It thus becomes evident that we can conceive a deterministic (unit) reference point and that the stochastic state of nature which is relevant for the agent is given by the value of X_{t+1}. It is also clear that the agent evaluates its unitary gain/loss $(X_{t+1}-1)$ by taking into account the possible future changes in prices and then multiplies this discounted unitary gain/loss by his activity (production) level L_t. His evaluation entails also, of course, the decreasing marginal utility represented by α.

The optimization problem of the representative agent on a typical island j can then be written as:

$$\max_{L_t} E_t(U_t) = \frac{L_t^\alpha}{\alpha}\left[E_t\left(\frac{P_t X_{t+1}}{P_{t+1}}\right)^\alpha + \beta E_t \hat{v}\right] - \frac{L_t^\psi}{\psi} \tag{4}$$

and we can state the following:

Proposition 1 *If: **i)** the equilibrium price function has the same general form as that of Lucas 1972, i.e., the equilibrium price at time t depends on the current state of the economy, $P_t(M_{t-1}, X_t, N_t)$, regardless of the route by which the equilibrium was attained; **ii)** the agents, in forming their rational expectations on the equilibrium price, adopt the same conjecture (which is verified in equilibrium) on $P_t(M_{t-1}, X_t, N_t)$ as that of Bénassy's 1999; then there will exist a range of economically meaningful parameters values such that the optimization problem (4) has a unique solution.*

Proof. In the island market equilibrium, the price of output is equal to:

$$P_t = \left(\frac{X_t}{N_t}\right)\frac{M_{t-1}}{L_t} = Z_t M_{t-1} L_t^{-1} \tag{5}$$

where $Z_t = X_t/N_t$ is known to the agent. The agent may thus adopt the following conjecture for the final equilibrium price:

$$P_t = \delta Z_t^\gamma M_{t-1} \tag{6}$$

[10]For a discussion of the properties of power utility specifications in prospect theory, see Köbberling and Wakker (2005).

where δ and γ are unknown coefficients to be determined. Substituting the conjecture (6) into the optimization problem (4) and recalling that $X_t = M_t / M_{t-1}$, we write:

$$\max_{L_t} E_t\left(U_t\right) = \frac{L_t^{\alpha}}{\alpha} E\left[\left(Z_t^{\gamma} \frac{N_{t+1}^{\gamma} X_{t+1}^{1-\gamma}}{X_t}\right)^{\alpha} |Z_t\right] \tag{7}$$

$$+\frac{\beta L_t^{\alpha}}{\alpha} E\left[\left\{\begin{array}{ll} \left(Z_t^{\gamma} \frac{N_{t+1}^{\gamma} X_{t+1}^{-\gamma}}{X_t}\right)^{\alpha}\left(X_{t+1}-1\right)^{\alpha} & \text{for } X_{t+1} \geq 1 \\ -\theta\left(Z_t^{\gamma} \frac{N_{t+1}^{\gamma} X_{t+1}^{-\gamma}}{X_t}\right)^{\alpha}\left[-\left(X_{t+1}-1\right)\right]^{\alpha} & \text{for } X_{t+1}<1 \end{array}\right. |Z_t\right] - \frac{L_t^{\psi}}{\psi}$$

Since N_{t+1}, X_t and X_{t+1} are independent variables, and the conditioning is relevant only for the computation of the expected value of X_t, the first expected term is:

$$E\left[\left(Z_t^{\gamma} N_{t+1}^{\gamma} X_{t+1}^{1-\gamma} X_t^{-1}\right)^{\alpha} |Z_t\right] = Z_t^{(\gamma-\rho)\alpha} \cdot e^{\frac{1}{2}\alpha^2\left(\rho+\gamma^2\right)\sigma_n^2+\frac{1}{2}\alpha^2(1-\gamma)^2\sigma_x^2} \tag{8}$$

with $\rho = \sigma_x^2 / \left(\sigma_x^2 + \sigma_n^2\right)$. The second expected term is equal to:[11]

$$E_t \hat{v} = Z_t^{(\gamma-\rho)\alpha} \cdot e^{\frac{1}{2}\alpha^2\left(\rho+\gamma^2\right)\sigma_n^2} \cdot \Gamma \tag{9}$$

where:

$$\Gamma = \int_1^{\infty}\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)^{\alpha} f_X dX_{t+1} + \int_0^1 -\theta\left[-\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)\right]^{\alpha} f_X dX_{t+1} \tag{10}$$

and f_X is the log-normal probability density of X_{t+1}. By substituting equations (8) and (9) into the agent's problem (7) we then obtain:

$$\max_{L_t} E_t\left(U_t\right) = Z_t^{(\gamma-\rho)\alpha} \frac{L_t^{\alpha}}{\alpha} \Lambda - \frac{L_t^{\psi}}{\psi} \tag{11}$$

where $\Lambda = e^{\frac{1}{2}\alpha^2\left(\gamma^2+\rho\right)\sigma_n^2}\left(e^{\frac{1}{2}\alpha^2(1-\gamma)^2\sigma_x^2} + \beta\Gamma\right)$. For problem (11) to have a unique and economically meaningful solution it must be $\Lambda > 0$. This condition is certainly verified when $\Gamma > 0$, but the computational analysis carried out in section 5 shows that it is $\Gamma < 0$ and $\Lambda > 0$ for a wide range of (economically reasonable) parameters' values. If $\beta = 0$, then $\Lambda_{\mathrm{L}} = e^{\left[\left(\gamma^2+\rho\right)\sigma_n^2+(1-\gamma)^2\sigma_x^2\right]\alpha^2/2}$ and the model collapses to the standard case. From the first order condition of problem (11) we get $L_t = \Lambda^{\frac{1}{\psi-\alpha}} Z_t^{\frac{(\gamma-\rho)\alpha}{\psi-\alpha}}$ and

$$P_t = \Lambda^{-\frac{1}{\psi-\alpha}} Z_t^{1-\frac{(\gamma-\rho)\alpha}{\psi-\alpha}} M_{t-1} \tag{12}$$

so that by equating (6) and (12) it follows that:

$$\delta = \Lambda^{-\frac{1}{\psi-\alpha}} \quad \text{and} \quad \gamma = 1 - \frac{\alpha}{\psi}(1-\rho) \tag{13}$$

[11] See appendix - sec. 1.

with $0 < \gamma < 1$. For any given configuration of the parameters' values, Γ and hence Λ and δ can be computed together with γ. This implies that the conjecture (6) is verified and that the unknown coefficients δ and γ can be determined.[12] ■

The conjecture (6) has the same general form as that the agent would make in the absence of v, but Λ (i.e., δ) is different from the case in which $\beta = 0$, whereas γ is the same. This is the analytical rationale of our subsequent findings: under rational expectations (given our overlapping generation framework) the presence of loss aversion leads the agents to correctly predict (on average) the price level, as it occurs in the case of $\beta = 0$; the *average* value of the equilibrium price is however different. This difference is measured by the term Γ which encapsulate the effect of the loss aversion. Figure 1 depicts different shapes of the integrand functions in Γ, i.e., the functions $\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)^{\alpha} f\left(X_{t+1}\right)$ for $X_{t+1} \geq 1$ and $-\theta\left[-\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)\right]^{\alpha} f\left(X_{t+1}\right)$ for $X_{t+1} < 1$, for three different values of the standard deviation σ_x.

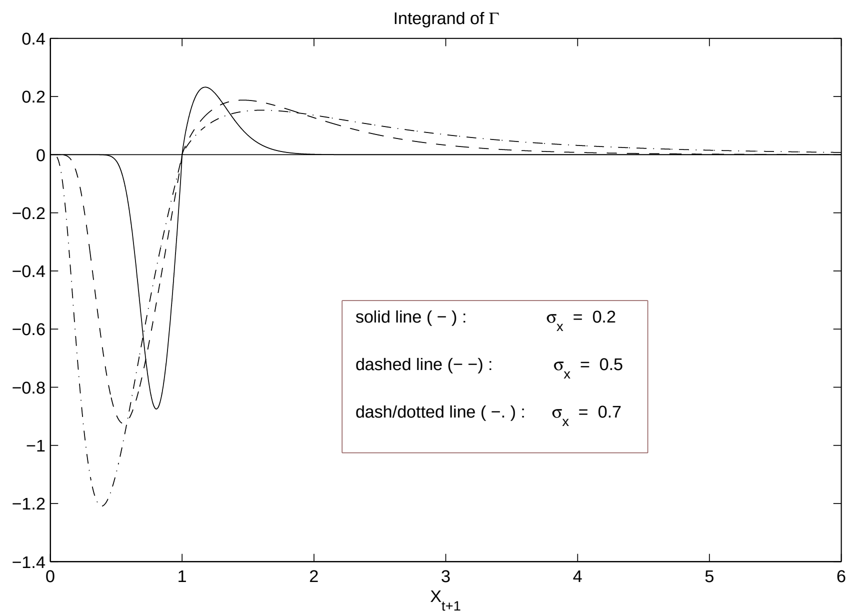

Figure 1: **The integrand of** Γ. The displayed functions have been drawn for the following values of the model's parameters: $\alpha = 0.88$, $\psi = 1$, $\theta = 2.25$ $\beta = 1$ and $\sigma_n^2 = 1$. These are the values of one of the parameterisations discussed in section 5.

Given the equilibrium price provided by equation (12), the equilibrium levels of labor

[12]Note that Propostion 1 determines a set of *sufficient* conditions for a closed form solution, in line with the equilibrium concept originally developed by Lucas.

and consumption for the whole island j can be determined:

$$L_t = \Lambda^{\frac{1}{\psi-\alpha}} \left(\frac{X_t}{N_t}\right)^{1-\gamma}; \qquad C_t = \Lambda^{\frac{1}{\psi-\alpha}} \frac{X_t^{1-\gamma} N_t^{\gamma}}{N_{t-1}} \tag{14}$$

3 Macroeconomic equilibrium

In equilibrium, the overall production level of each market (island) j is $Y_{j,t}^M = N_{j,t} L_{j,t} = N_{j,t} \Lambda^{\frac{1}{\psi-\alpha}} \left(\frac{X_t}{N_{j,t}}\right)^{1-\gamma}$ and the economy-wide level of output is $Y_t = \left(\sum_j Y_{j,t}^M\right)/J$ where J is the number of islands. The island population $N_{j,t}$ is equal to $N_{j,t} = \Xi_{j,t} N_t$, where N_t is the overall level of population and Ξ_{jt} is an i.i.d. shock log-normally distributed in all the islands, with $\xi_{j,t} = \ln \Xi_{j,t} \sim N\left(0, \sigma_\xi^2\right)$. If, in addition, N_t is also a log-normal random variable with distribution $n_t = \ln N_t \sim N\left(0, \sigma_N^2\right)$, we only need to substitute $\sigma_n^2 = \sigma_\xi^2 + \sigma_N^2$ in the analysis of the previous section in order to confirm all our results.

Recalling that X_t is a common shock, the aggregate level of output can be written as:

$$Y_t = N_t^{\gamma} \Lambda^{\frac{1}{\psi-\alpha}} X_t^{1-\gamma} \left(\sum_j \Xi_{j,t}^{\gamma}\right)/J$$

If we assume that the number of islands is large enough, by applying the law of large numbers we may write $\sum_j \Xi_{j,t}^{\gamma}/J = E\left(\Xi_{j,t}^{\gamma}\right) = \exp\left(\gamma^2 \sigma_\xi^2/2\right)$. It follows that $Y_t = \Omega N_t^{\gamma} X_t^{1-\gamma}$, where $\Omega = e^{\gamma^2 \sigma_\xi^2/2} \Lambda^{\frac{1}{\psi-\alpha}}$, or in logs:[13]

$$y_t = (1-\gamma) x_t + \gamma n_t + \omega \tag{15}$$

As for the price level, we may write: $P_t = \left(\sum_j Y_{jt}^M P_{jt}\right)/\sum_j Y_{jt}^M$. Using the market equilibrium condition for the j-th island, $X_t M_{j,t-1} = P_{j,t} Y_{j,t}^M = N_{j,t} P_{j,t} L_{j,t}$, we have: $P_t = X_t M_{t-1}/Y_t$. Substituting (15) into this equation we obtain $P_t = \left(\frac{X_t}{N_t}\right)^{\gamma} \frac{M_{t-1}}{\Omega}$, so that the inflation rate (in log-values) is equal to:

$$\pi_t = p_t - p_{t-1} = \gamma x_t + (1-\gamma) x_{t-1} + \gamma (n_{t-1} - n_t) \tag{16}$$

Equations (15)-(16) synthesize the first result of our model: inflation dynamics and output fluctuations - i.e., the equilibrium response of macroeconomic variables to unexpected shocks - are not affected by the agents' loss aversion, which only modifies, *via* ω, the *average* level of output and prices. In principle, the introduction of loss averse agents may vary the aggregate level of potential output below ($\Lambda < \Lambda_{\mathrm{L}}$) or above ($\Lambda > \Lambda_{\mathrm{L}}$) that obtained in

[13]Equation (15) simply restate a well-known textbook result related to the signal extraction model: output fluctuations with respect to its average (or "natural") level are an equilibrium phenomenon due to both real and monetary shocks. If we use the dynamic money supply rule $M_t = X_t M_{t-1}$ at an aggregate level, after some simple manipulation the output dynamics (15) can be rewritten as $y_t = (1-\gamma)[m_t - E(m_t)] + \gamma n_t + \omega$, so that the only distrurbances in the money supply capable of affecting real output are those which are unforeseen by the agents.

the standard case. In section 5 we show however that the first case is the one that obtains for the ranges of parameters' values identified by the literature: the agents' loss aversion decreases potential output.

When formulating their forecasts on the economy's evolution, agents clearly consider the possibility that monetary shock X_{t+1} induce gains or losses with respect to their reference wealth. Given that their rational expectations will be (on average) fulfilled, they take into account this effect when formulating a lower average supply of labour (and/or demand of consumption) as a form of precautionary behavior. The difference in the levels of Λ and ω with respect to the case $\beta = 0$ may thus be interpreted as an equilibrium insurance premium to be paid against the possibility of experiencing deviations of wealth with respect to its reference amount. This implies that the agents react to unexpected shocks disregarding the gains/losses induced by the monetary shocks.

This result has the welcome effect to preserve most of the desirable features of the original signal extraction model which are related to the cyclical behavior of y_t and π_t, such as: i) the positive correlation of y_t and π_t with x_t; ii) the possibility of a negative covariance between y_t and π_t due to the presence of real shocks (i.e., when $\sigma_n^2 > 0$); iii) the non-linear relationship between the output variance σ_y^2 and the monetary variance σ_x^2 commonly found in the data.

4 Welfare implications

We now investigate the consequences on welfare of the variability of monetary policy. Clearly, the introduction of behavioral elements in macroeconomic models makes the definition of an adequate aggregate welfare indicator more complex. For instance, it is not clear wether rational policy-makers should construct the social welfare function starting from the actual, and "not fully rational", agents' utility functions, or should instead adhere to the tenets of standard expected utility theory.[14] In the face of this open debate, and in line with previous versions of structural signal extraction models, we choose to focus on the representative agent's equilibrium level of welfare. This is also coherent with the agreement reached in the more recent macroeconomic literature on the possibility to identify the social welfare function with the expected utility of the representative agent.[15]

We then compute the expected utility of an agent who may end up in a generic island of dimension Ξ_t:

$$W = \frac{E\left(\Xi_t \cdot U_t\right)}{E\left(\Xi_t\right)} = \frac{E\left(\Xi_t \frac{C_{t+1}^{\alpha}}{\alpha}\right) - E\left(\Xi_t \frac{L_t^{\psi}}{\psi}\right) + \frac{\beta}{\alpha} E\left[\Xi_t v\left(R_{t+1}, R_{\text{ref}}\right)\right]}{E\left(\Xi_{j,t}\right)} \tag{17}$$

and focus on Lucas' (1972) original case in which the real shocks are of a purely relative nature ($\sigma_N^2 = 0$), so that the overall population can be normalized to one ($N_{j,t} = \Xi_{j,t}$). Substituting the equilibrium values of C and L (equations (14)) into equation (17), disregarding

[14]See Dhami and al Nowaihi (2010).

[15]See, for instance, Benigno and Woodford (2005).

for simplicity the index j, and recalling that the variables are statistically independent, we obtain:[16]

$$W = K \cdot W_{\mathrm{L}} \qquad \text{or also:} \qquad W = \left(\frac{\Lambda}{\Lambda_{\mathrm{L}}}\right) W_{\mathrm{L}} \tag{18}$$

where $K = \Lambda/\Lambda_{\mathrm{L}} = \left(1 + \beta e^{-\frac{1}{2}\alpha^2(1-\gamma)^2\sigma_x^2}\Gamma\right)^{\frac{\psi}{\psi-\alpha}}$ and W_{L}, the welfare level that would be obtained in the absence of the PT component (i.e. when $\beta = 0$), is equal to:[17]

$$\begin{aligned} W_{\mathrm{L}} &= \frac{1}{\alpha}\exp\left\{\frac{\alpha\sigma_n^2}{2}\left[\frac{\alpha^2(3\psi-\alpha)}{(\psi-\alpha)\psi}\rho - \frac{\alpha^2}{\psi} - 2(1-\alpha)\right]\right\} \\ &\quad -\frac{1}{\psi}\exp\left\{\frac{\alpha\sigma_n^2}{2}\left[\frac{\alpha^2(3\psi-\alpha)}{(\psi-\alpha)\psi}\rho + 2\rho - \frac{\alpha^2}{\psi} - 2(1-\alpha)\right]\right\} \end{aligned}$$

Thus it is $W < W_{\mathrm{L}}$ as long as $W_{\mathrm{L}} > 0$ and $\Gamma < 0$ (for the range of parameters' values guaranteeing $\Lambda > 0$): the PT component has a negative impact on average welfare, by proportionally reducing it by a factor K. If $W_{\mathrm{L}} < 0$ it is instead $W > W_{\mathrm{L}}$ for $\Gamma < 0$ and $\Lambda > 0$.

The reaction of W to changes in the variance of monetary policy can be studied by taking ρ as the policy variable (and keeping σ_n^2 constant). We obtain:[18]

$$\frac{\partial W}{\partial \rho} = K\frac{\partial W_{\mathrm{L}}}{\partial \rho} + \frac{\partial K}{\partial \rho}W_{\mathrm{L}}$$

and

$$\frac{\partial K}{\partial \rho} = \frac{\beta\psi}{\psi-\alpha}HK^{\frac{\alpha}{\psi}}\left[\frac{\partial \Gamma}{\partial \rho} - \frac{1}{2}\frac{\alpha^4}{\psi^2}(1-2\rho)\sigma_n^2\Gamma\right]$$

where $H = e^{-\frac{1}{2}\frac{\alpha^4}{\psi^2}\rho(1-\rho)\sigma_n^2}$. The derivative $\partial\Gamma/\partial\rho$ is equal to:[19]

$$\frac{\partial \Gamma}{\partial \rho} = \int_1^{\infty} G_X\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)^{\alpha} f_X dX_{t+1} - \theta\int_0^1 G_X\left[-\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)\right]^{\alpha} f_X dX_{t+1}$$

and G_X is a function of X_{t+1} and ρ:

$$G_X = -\frac{\alpha^2}{\psi}\ln(X_{t+1}) + \frac{(\ln X_{t+1})^2}{2\rho^2\sigma_n^2} - \frac{1}{2\rho(1-\rho)}$$

Since the term $\partial\Gamma/\partial\rho$ cannot be solved analytically, in the next section we resort to a computational analysis in order to understand how W changes w.r.t. σ_x.

[16] See appendix - sec. 2.

[17] See Bénassy (1999).

[18] Where

$$\frac{\partial W_{\mathrm{L}}}{\partial \rho} = B_{\mathrm{L}}\left\{\frac{\alpha^2(3\psi-\alpha)}{\psi(\psi-\alpha)} - \left[\frac{\alpha^3(3\psi-\alpha)}{\psi^2(\psi-\alpha)} + \frac{2\alpha}{\psi}\right]\exp\left[\alpha\sigma_n^2\rho\right]\right\}$$

and

$$B_{\mathrm{L}} = \frac{\sigma_n^2}{2}\exp\frac{\alpha\sigma_n^2}{2}\left[\frac{\alpha^2(3\psi-\alpha)}{(\psi-\alpha)\psi}\rho - \frac{\alpha^2}{\psi} - 2(1-\alpha)\right]$$

.

[19] This expression is obtained by using the Leibniz rule of differentiation in equation (10).

5 Computational analysis

The presence of Γ and $\partial\Gamma/\partial\rho$ requires to investigate the magnitude of Λ, Γ, W and W_{L} for different parameters' values by using a numerical version of our model. To this aim we fix only two coefficients: $\theta = 2.25$ and $\sigma_n^2 = 1$. The first one derives from a wide amount of experimental evidence (see, e.g., Tversky and Kahneman 1992). The second one allows us to measure the variability of monetary policy, σ_x^2, as a proportion of real uncertainty σ_n^2.

Due to the conflicting evidence on α and ψ, we consider instead a wide range of values for each of these parameters. As for the curvature coefficient we fix the upper value of its interval at $\alpha = 0.88$, which is commonly accepted from experimental evidence in behavioral economics (Tversky and Kahneman 1992), and the lower bound at $\alpha = 0.26$. This is the average between 0.88 and -0.35, the latter being the value obtained for the richest specification of recent estimations (Engelhardt and Kumar 2009).[20] As for ψ, we choose the upper bound $\psi = 10$, which is supported by widely accepted microeconomic estimates (Card 2004; Trigari 2009), and the lover bound $\psi = 1$, which is the most commonly adopted value in macroeconomic studies. In the absence of strong priors and noting that β does not affect the *sign* of Γ but only that of Λ, we adopt the benchmark value $\beta = 1$ (a unitary financial loss provides the same disutility as a unit of foregone consumption) and perform robustness checks of our results by changing the value of this coefficient. Finally, we let σ_x vary between 0.01 (one standard deviation) and the value for which $\Lambda = 0$.

Since the qualitative results remain the same for the other parameters' values contained in the chosen intervals, we present here only the evidence that obtains in the four cases represented by the combinations of the extreme values of α and ψ. Figure 2 below summarizes our numerical results for the values of Γ and Λ.

The computational exercise supports the conclusions that, for a wide range of parameters' values consistent with the empirical evidence: i) the claim made in proposition 1 is correct; ii) the introduction of loss aversion reduces the natural level of output ($\Lambda < \Lambda_{\mathrm{L}}$). This conclusion is robust with respect to changes in β: an increase in β only negatively affects the value of σ_x for which it is $\Lambda > 0$. This means that increases in the weight of the loss aversion component in utility monotonically reduce the maximum acceptable value of monetary variability as compared to real variability.[21]

Figure 3 shows an analogous exercise for the values of W and W_{L}. We can summarize our results relative to the welfare impact of loss aversion in two main observations.

First, as hinted at in section 4, expected welfare W is lower than in the standard model ($\beta = 0$) when $W_{\mathrm{L}} > 0$, and this is the case under most of the economically acceptable parameterizations. In some instances (e.g., the case $\alpha = 0.88$ and $\psi = 1$), the behavior of W is instead more complex. For small σ_x welfare W is lower than W_{L}, but as σ_x increases W_{L} becomes negative and is overcome by W. The change of sign in W_{L} occurs when the disutility of work, $-\frac{1}{\psi}\exp\left\{\frac{\alpha\sigma_n^2}{2}\left[\frac{\alpha^2(3\psi-\alpha)}{(\psi-\alpha)\psi}\rho + 2\rho - \frac{\alpha^2}{\psi} - 2(1-\alpha)\right]\right\}$, becomes greater (in modulus) than the utility generated by consumption, represented by

[20] Engelhardt and Kumar (2009) estimate a range of values for α between 0.17 and -1.7. Yet, recall that in our model we need $\alpha > 0$ for coherence with the power utility specifications adopted in prospect theory.

[21] For example, with $\alpha = 0.88$ and $\psi = 10$, for having $\Lambda \simeq 0$ it must be $\sigma_x = 1.19$ when $\beta = 0.5$ and $\sigma_x = 0.53$ when $\beta = 2$.

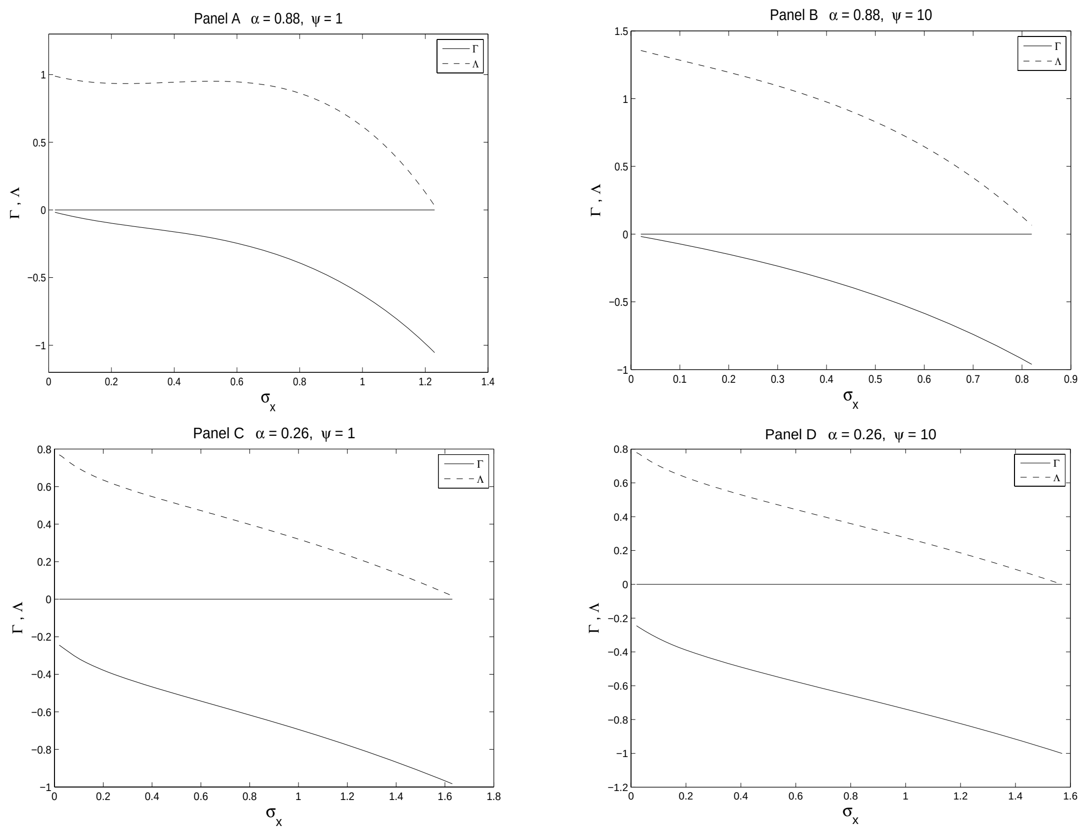

Figure 2: **Parameters' range for model consistency**. The four panels show the behaviour of Γ and Λ for different values of σ_x, starting from a value of the monetary shock standard deviation close to 0. The upper limit of σ_x in each panel is set to the largest value for which $\Lambda \geq 0$.

$\frac{1}{\alpha}\exp\left\{\frac{\alpha\sigma_n^2}{2}\left[\frac{\alpha^2(3\psi-\alpha)}{(\psi-\alpha)\psi}\rho - \frac{\alpha^2}{\psi} - 2\left(1-\alpha\right)\right]\right\}$. This is more likely to occur when α is close to ψ and hence, under this conditions, working less improves utility. Since loss aversion induces agents to supply less labor, welfare is higher than W_{L}: this explains the relative position of W and W_{L} in panel A. Furthermore, notice that when σ_x is high enough then Λ, K and W tend to zero, and this justifies the inverted humped-shaped behavior of W in Panel A. Welfare initially decreases and becomes negative but, from a certain value of σ_x onward, factor K shrinks and drives W towards zero.

Second, the inclusion of loss aversion in the structural signal extraction model eliminates the paradoxical effect of σ_x on welfare highlighted by Polemarchakis and Weiss (1977) and Bénassy (1999). They show that, starting from $\rho = 0$, an increase in σ_x improves welfare when $\alpha < 0$, or when $\alpha > 0$ and $\psi\left(2\alpha - 1\right) - \alpha^2 > 0$.[22] By inspecting figure 3, it is evident

[22] This is due to the fact that, under this specific parameters' configuration, an increase in monetary uncertainty induces private agents to react to real uncertainty in such a way as to choose a level of their

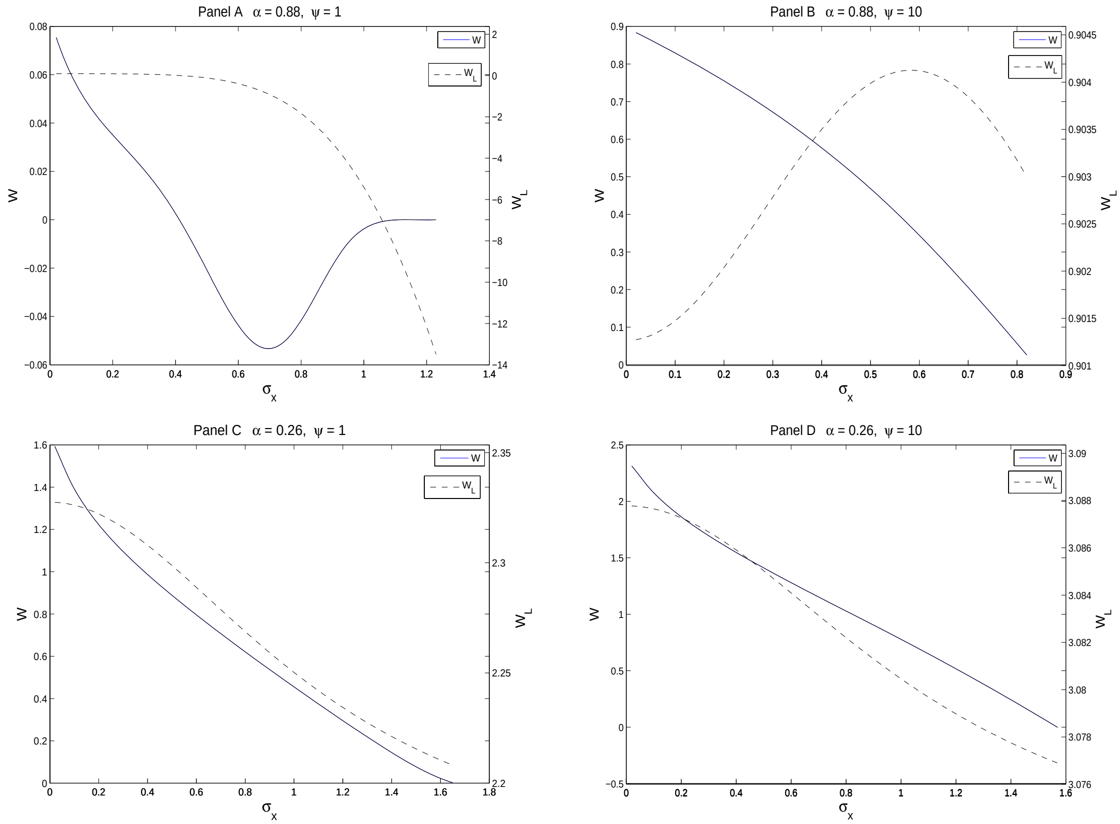

Figure 3: **Welfare response to monetary uncertainty**. The four panels show the behaviour of W and W_{L} for the same ranges of σ_x as those adopted in Figure 2. The graphs are in double scale: the right vertical axis shows the values of W_{L}. The upper limit of σ_x in each panel is again set to the largest value for which $\Lambda \geq 0$.

that this "pathological" reaction of welfare to monetary uncertainty can never occur in our model. In particular, this is true also in the case $\alpha = 0.88$ and $\psi = 1$, because here the maximum level of W is obtained when σ_x tends to zero, as it can be seen in Panel A.[23] Lucas's prescription $-$ as far as the surprise inflation is concerned $-$ is thus fully restored under loss aversion.

The explanation of this second result runs as follows. Bénassy (1999) points out that, in his model and for some parameterizations, a more random monetary policy (an increase in σ_x) brings the agents' reactions to N_t closer to that obtained by solving the social planner's problem, i.e., by maximizing $E\left(\Xi_t \cdot U_t\right)$, where U_t of course excludes the the PT

labour supply which is closer to the one that would be chosen by a paretian planner.

[23]Note however that the U-shaped behavior of W in Panel A of figure 3 is limited to a relatively small set of parameters' values. For instance, this behaviour is present, if $\alpha = 0.88$, when $\psi \in [1; 1.5]$ and, if $\psi = 1$, when $\alpha \in [0.88; \cong 0.71]$.

component, subject to the feasibility constraints. The resulting equation for the optimal quantity of labor is:

$$L_t^* = \left(\frac{N_{t-1}}{N_t}\right)^{(1-\alpha)/(\psi-a)}$$

which is to be compared to the value found for the market equilibrium:

$$L_{\mathrm{L},t} = \Lambda_{\mathrm{L}}^{\frac{1}{\psi-\alpha}} \left(\frac{X_t}{N_t}\right)^{(1-\rho)\frac{\alpha}{\psi}}$$

where $\Lambda_{\mathrm{L}} = e^{\left[(\gamma^2+\rho)\sigma_n^2+(1-\gamma)^2\sigma_x^2\right]\alpha^2/2}$. An increase in ρ (or σ_x) moves the exponent of the market labor supply closer to that of L_t^*. A more volatile monetary policy may hence improve welfare (for some values of α and ψ) up to a maximum level.

By adopting the same criterion, in our model we clearly obtain the same optimal value for L_t^*, but the market solution is different:

$$L_t = \left[\Lambda_{\mathrm{L}} + e^{\frac{1}{2}\alpha^2\left[\left(1-\frac{\alpha}{\psi}(1-\rho)\right)^2+\rho\right]\sigma_n^2}\beta\Gamma\right]^{\frac{1}{\psi-\alpha}} \left(\frac{X_t}{N_t}\right)^{(1-\rho)\frac{\alpha}{\psi}}$$

Now an increase in ρ has the same effect on the exponent of the market labor supply as in Bénassy (1999) but it produces a different impact on the coefficient Λ (an element disregarded in his discussion) since it raises the absolute value of $e^{\frac{1}{2}\alpha^2\left[\left(1-\frac{\alpha}{\psi}(1-\rho)\right)^2+\rho\right]\sigma_n^2}\beta\Gamma$ which, having a negative sign ($\Gamma < 0$), counteracts the increase in Λ_{L}. The presence of a PT component thus prevents the paradoxical effect produced by a rise in σ_x: although the reaction to N_t due to the exponent is closer to that prescribed by the social planner, loss adverse agents tend to be more "cautious" than standard expected utility maximizers, the more so the higher is the volatility of monetary policy.

6 Conclusions

By modifying a structural signal extraction model so as to take into account some behavioral features, in this paper we have studied a monetary islands economy inhabited by agents exhibiting reference dependence, declining sensitivity and loss aversion when evaluating their financial assets. Our analysis suggests that potential output is affected not only by market frictions, but also by "behavioral frictions". More in particular, we have shown that the presence of loss averse agents lowers potential output for a wide range of empirically accepted parameters' values. This is so because their expectations that the monetary resources brought to the next period may be affected (in real terms) by a monetary shock hitting the economy induce them to decrease their equilibrium labor supply. This finding is somehow reminiscent of Keynes' intuition that, in a world where money is a storage of value, agents' psychological attitudes may constitute a channel through which financial phenomena impact the economy's real side.

From the aggregate welfare perspective, we have shown that, for most of the parameterisations we have considered, the representative agent's expected utility under loss aversion is lower than in the standard versions of the model. At the normative level, our analysis could hence suggest to increase the potential level of output and welfare by introducing (or favouring through policy incentives) financial assets preventing agents from making losses, like capital guaranteed funds or inflation-indexed fixed-income pension funds.

These remarks notwithstanding, we of course consider the model presented here only as a first step towards a more thorough inclusion of behavioral economics into microfuonded macroeconomic models. An interesting avenue for future research could be - among others - to explore the implications of our model for monetary policy games and the related issue of central bank transparency.

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Appendix

1. *Derivation of equation (9).*

In order to compute $E_t\hat{v}$, we insert the conjecture $P_t = \delta Z_t^{\gamma} M_{t-1}$ into $E_t\hat{v}$, and by recalling that it is $Z_t = X_t/N_t$ and $X_t = M_t/M_{t-1}$,we get:

$$E_t\hat{v} = Z_t^{\alpha\gamma} E_t \left[\left(\frac{N_{t+1}^{\gamma}}{X_{t+1}^{\gamma} X_t} \right)^{\alpha} \begin{cases} (X_{t+1} - 1)^{\alpha} & \text{for } X_{t+1} - 1 \geq 0 \\ -\theta \left[-(X_{t+1} - 1) \right]^{\alpha} & \text{for } X_{t+1} - 1 < 0 \end{cases} \right]$$

This conditional expected value in the square bracket requires to perform an integration of two distinct "branches", one for the case of loss and the other one for the case of gains:

$E_t\left(N_{t+1}, X_{t+1}, X_t\right) =$

$$\begin{cases} \int_{N_{t+1}} \int_{X_t} \int_{X_{t+1}} \frac{N_{t+1}^{\alpha\gamma}\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)^{\alpha}}{X_t^{\alpha}} F_Z dN_{t+1} dX_{t+1} dX_t & \text{for } X_{t+1} - 1 \geq 0 \\ -\theta \int_{N_{t+1}} \int_{X_t} \int_{X_{t+1}} \frac{N_{t+1}^{\alpha\gamma}\left[-\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)\right]^{\alpha}}{X_t^{\alpha}} F_Z dN_{t+1} dX_{t+1} dX_t & \text{for } X_{t+1} - 1 < 0 \end{cases}$$

where $F_Z = F\left(N_{t+1}, X_{t+1}, X_t \,|Z_t\right)$ is the joint distribution of the three variables. Note however that N_{t+1}, X_{t+1} and X_t are independent (and the conditioning acts only on X_t) so that we can write:

$E_t\left(N_{t+1}, X_{t+1}, X_t\right) =$

$$\begin{cases} \int N_{t+1}^{\alpha\gamma} f\left(N_{t+1}\right) dN_{t+1} \int X_t^{-\alpha} f\left(X_t \,|Z_t\right) dX_t \int \left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)^{\alpha} f\left(X_{t+1}\right) dX_{t+1} & \text{for } \quad X_{t+1} - 1 \geq 0 \\ -\theta \int N_{t+1}^{\alpha\gamma} f\left(N_{t+1}\right) dN_{t+1} \int X_t^{-\alpha} f\left(X_t \,|Z_t\right) dX_t \int \left[-\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)\right]^{\alpha} f\left(X_{t+1}\right) dX_{t+1} & \text{for } \quad X_{t+1} - 1 < 0 \end{cases}$$

where f is the log-normal (univariate) distribution. The expression is still split into two different branches, but we know that it is only the term $\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)$ that determines the gain/loss situation. Furthermore, due to statistical independence, the other two functions of N_{t+1} and X_t must be integrated under their respective variables alone. Hence the two split branches refers to integration of X_{t+1} over two separated intervals - respectively $[0, 1)$ and $(1, +\infty)$ - while the other two integrals (those w.r.t. N_{t+1} and X_t) must be integrated over the full interval: $N_{t+1}, X_t \in [0, +\infty)$ in both branches.

We can thus sum (under integration) the two branches in the following way:

$E_t\left(N_{t+1}, X_{t+1}, X_t\right) =$

$$\int_{N_{t+1}=0}^{N_{t+1}=\infty} N_{t+1}^{\alpha\gamma} f\left(N_{t+1}\right) dN_{t+1} \int_{X_t=0}^{X_t=\infty} X_t^{-\alpha} f\left(X_t \,|Z_t\right) dX_t \int_{X_{t+1}\in[1,\infty)} \left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)^{\alpha} f\left(X_{t+1}\right) dX_{t+1} +$$

$$\int_{N_{t+1}=0}^{N_{t+1}=\infty} N_{t+1}^{\alpha\gamma} f\left(N_{t+1}\right) dN_{t+1} \int_{X_t=0}^{X_t=\infty} X_t^{-\alpha} f\left(X_t \,|Z_t\right) dX_t \int_{X_{t+1}\in[0,1)} -\theta \left[-\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)\right]^{\alpha} f\left(X_{t+1}\right) dX_{t+1}$$

so that the integrals in N_{t+1} and X_t can be factored out:

$$\int_0^\infty N_{t+1}^{\alpha\gamma} f\left(N_{t+1}\right) dN_{t+1} \int_0^\infty X_t^{-\alpha} f\left(X_t \left| Z_t\right.\right) dX_t \left\{ \begin{array}{c} \int_{X_{t+1}\in[1,\infty)} \left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)^\alpha f\left(X_{t+1}\right) dX_{t+1} \\ -\theta \int_{X_{t+1}\in[0,1)} \left[-\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)\right]^\alpha f\left(X_{t+1}\right) dX_{t+1} \end{array} \right\}$$

The two expressions $\int_0^\infty N_{t+1}^{\alpha\gamma} f\left(N_{t+1}\right) dN_{t+1}$ and $\int_0^\infty X_t^{-\alpha} f\left(X_t \left| Z_t\right.\right) dX_t$ are respectively equal to $E\left(X_t^{-\alpha} \left| Z_t\right.\right) = e^{\frac{1}{2}\alpha^2\gamma^2\sigma_n^2}$ and $E\left(N_{t+1}^{\gamma\alpha}\right) = Z_t^{-\alpha\rho} e^{\frac{1}{2}\alpha^2\rho\sigma_n^2}$, so that we have:

$$Z_t^{-\alpha\rho} e^{\frac{1}{2}\alpha^2\left(\rho+\gamma^2\right)\sigma_n^2} \cdot \left\{\int_1^\infty \left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)^\alpha f\left(X_{t+1}\right) dX_{t+1} + \int_0^1 -\theta\left[-\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)\right]^\alpha f\left(X_{t+1}\right) dX_{t+1}\right\}$$

where the sum in the braces is equal to Γ in equation (10).

2. *Derivation of equation (18).*

In order to compute $W = E\left(\Xi_t \cdot U_t\right)/E\left(\Xi_t\right)$, the following procedure must be adopted. First the term $E\left(\Xi_t \cdot U_t\right)$ must be computed, where the expected value is only an approximation due to the application of the law of great numbers (this means that the expectation is unconditioned). Second, the resulting expression for $E\left(\Xi_t \cdot U_t\right)$ must be divided by $E\left(\Xi_t\right) = \exp\left(\frac{1}{2}\sigma_n^2\right)$.

From equation (12) we know that it is: $P_t/P_{t+1} = Z_t^\gamma N_{t+1}^\gamma X_{t+1}^{-\gamma}/X_t$. Hence, by imposing $N_t = \Xi_t$ together with equations (14), we can compute:

$$\begin{aligned}
\left(\Xi_t \cdot U_t\right) &= \frac{1}{\alpha}\Lambda^{\frac{\alpha}{\psi-\alpha}} X_{t+1}^{(1-\gamma)\alpha} \Xi_{t+1}^{\alpha\gamma} \Xi_t^{1-\alpha} - \frac{1}{\psi}\Lambda^{\frac{\psi}{\psi-\alpha}} X_t^{(1-\gamma)\psi} \Xi_t^{1-(1-\gamma)\psi} \\
&\quad + \frac{\beta}{\alpha}\Lambda^{\frac{\alpha}{\psi-\alpha}} \Xi_{t+1}^{\alpha\gamma} \Xi_t^{1-\alpha} \left\{ \begin{array}{ll} \left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)^\alpha & \text{for } X_{t+1} \geq 1 \\ -\theta\left[-\left(X_{t+1}^{1-\gamma} - X_{t+1}^{-\gamma}\right)\right]^\alpha & \text{for } X_{t+1} < 1 \end{array} \right.
\end{aligned}$$

The average $E\left(\Xi_t \cdot U_t\right)$ can be computed by using the same procedure shown in the previous section:

$$E\left(\Xi_t \cdot U_t\right) = \frac{1}{\alpha}\Lambda^{\frac{\alpha}{\psi-\alpha}} E X_{t+1}^{(1-\gamma)\alpha} E\Xi_{t+1}^{\alpha\gamma} E\Xi_t^{1-\alpha} - \frac{1}{\psi}\Lambda^{\frac{\psi}{\psi-\alpha}} E X_t^{(1-\gamma)\psi} E\Xi_t^{1-(1-\gamma)\psi} + \frac{\beta}{\alpha}\Gamma\Lambda^{\frac{\alpha}{\psi-\alpha}} E\Xi_{t+1}^{\alpha\gamma} E\Xi_t^{1-\alpha}$$

where Γ, due to uncorrelation, is the same expression as in (10). The first and last terms of $E\left(\Xi_t \cdot U_t\right)$ can be grouped:

$$E\left(\Xi_t \cdot U_t\right) = \frac{1}{\alpha} e^{\frac{1}{2}\left[\alpha^2\gamma^2\sigma_n^2 + (1-\alpha)^2\sigma_n^2\right]} \Lambda^{\frac{\alpha}{\psi-\alpha}} \left(e^{\frac{1}{2}\left[\alpha^2(1-\gamma)^2\sigma_x^2\right]} + \beta\Gamma_\alpha\right) - \frac{1}{\psi}\Lambda^{\frac{\psi}{\psi-\alpha}} e^{\frac{1}{2}\left[\psi^2(1-\gamma)^2\sigma_x^2 + [1-\psi(1-\gamma)]^2\sigma_n^2\right]}$$

Note that, being $\Lambda = e^{\frac{1}{2}\alpha^2\left(\gamma^2+\rho\right)\sigma_n^2}\left(e^{\frac{1}{2}\alpha^2(1-\gamma)^2\sigma_x^2} + \beta\Gamma_\alpha\right)$, it is: $\left[e^{\frac{1}{2}\alpha^2(1-\gamma)^2\sigma_x^2} + \beta\Gamma_\alpha\right] = \Lambda e^{-\frac{1}{2}\alpha^2\left(\gamma^2+\rho\right)\sigma_n^2}$, which allows us to write:

$$E\left(\Xi_t \cdot U_t\right) = \frac{1}{\alpha} e^{\frac{1}{2}\left[\alpha^2\gamma^2\sigma_n^2 + (1-\alpha)^2\sigma_n^2\right]} e^{-\frac{1}{2}\alpha^2\left(\gamma^2+\rho\right)\sigma_n^2} \Lambda^{\frac{\alpha}{\psi-\alpha}} \Lambda - \frac{1}{\psi}\Lambda^{\frac{\psi}{\psi-\alpha}} e^{\frac{1}{2}\left[\psi^2(1-\gamma)^2\sigma_x^2 + [1-\psi(1-\gamma)]^2\sigma_n^2\right]}$$

Hence $E\left(\Xi_t \cdot U_t\right)$ can also be written as:

$$E\left(\Xi_t \cdot U_t\right) = \Lambda^{\frac{\psi}{\psi-\alpha}} \left\{ \frac{1}{\alpha} e^{\frac{1}{2}\left[(1-\alpha)^2\sigma_n^2 - \alpha^2\rho\sigma_n^2\right]} - \frac{1}{\psi} e^{\frac{1}{2}\left[\psi^2(1-\gamma)^2\sigma_x^2 + [1-\psi(1-\gamma)]^2\sigma_n^2\right]} \right\}$$

By using again $\Lambda^{\frac{\psi}{\psi-\alpha}} = e^{\frac{1}{2}\alpha^2\left(\gamma^2+\rho\right)\sigma_n^2\frac{\psi}{\psi-\alpha}} \left(e^{\frac{1}{2}\alpha^2(1-\gamma)^2\sigma_x^2} + \beta\Gamma_\alpha\right)^{\frac{\psi}{\psi-\alpha}}$ we have:

$$E\left(\Xi_t \cdot U_t\right) = \left(e^{\frac{1}{2}\alpha^2(1-\gamma)^2\sigma_x^2} + \beta\Gamma_\alpha\right)^{\frac{\psi}{\psi-\alpha}} \left\{ \begin{array}{c} \frac{1}{\alpha} e^{\frac{1}{2}\left[(1-\alpha)^2\sigma_n^2 - \alpha^2\rho\sigma_n^2 + \alpha^2\left(\gamma^2+\rho\right)\sigma_n^2\frac{\psi}{\psi-\alpha}\right]} \\ -\frac{1}{\psi} e^{\frac{1}{2}\left[\psi^2(1-\gamma)^2\sigma_x^2 + [1-\psi(1-\gamma)]^2\sigma_n^2 + \alpha^2\left(\gamma^2+\rho\right)\sigma_n^2\frac{\psi}{\psi-\alpha}\right]} \end{array} \right\}$$

We now can multiply and divide the right hand side by $\exp\left(\frac{1}{2}\alpha^2\left(1-\gamma\right)^2\sigma_x^2\frac{\psi}{\psi-\alpha}\right)$ and rearrange the expressions in the exponentials inside the braces, so to write:

$$E\left(\Xi_t \cdot U_t\right) = K \left\{ \begin{array}{l} \frac{1}{\alpha} e^{\frac{1}{2}\left\{\left[\alpha^2\gamma^2 + (1-\alpha)^2 + \frac{\alpha}{\psi-\alpha}\alpha^2\left(\gamma^2+\rho\right)\right]\sigma_n^2 + \left[\alpha^2(1-\gamma)^2\frac{\alpha}{\psi-\alpha} + \alpha^2(1-\gamma)^2\right]\sigma_x^2\right\}} \\ -\frac{1}{\psi} e^{\frac{1}{2}\left\{\left[\psi^2(1-\gamma)^2 + \alpha^2(1-\gamma)^2\frac{\psi}{\psi-\alpha}\right]\sigma_x^2 + \left[[1-\psi(1-\gamma)]^2 + \alpha^2\left(\gamma^2+\rho\right)\frac{\psi}{\psi-\alpha}\right]\sigma_n^2\right\}} \end{array} \right\}$$

where: $K = \left(1 + e^{-\frac{1}{2}\alpha^2(1-\gamma)^2\sigma_x^2}\beta\Gamma_\alpha\right)^{\frac{\psi}{\psi-\alpha}}$. If in the two exponentials in the braces the monetary variance is replaced by $\sigma_x^2 = \frac{\rho}{1-\rho}\sigma_n^2$, and recalling that it is $\gamma = 1 - \alpha\left(1-\rho\right)/\psi$, the same exponentials can be written in this way:

$$\frac{1}{\alpha}\exp\left\{\frac{\alpha\sigma_n^2}{2}\left[\frac{\alpha^2\left(3\psi-\alpha\right)}{\left(\psi-\alpha\right)\psi}\rho - \frac{\alpha^2}{\psi} - 2\left(1-\alpha\right)\right]\right\} - \frac{1}{\psi}\exp\left\{\frac{\alpha\sigma_n^2}{2}\left[\frac{\alpha^2\left(3\psi-\alpha\right)}{\left(\psi-\alpha\right)\psi}\rho + 2\rho - \frac{\alpha^2}{\psi} - 2\left(1-\alpha\right)\right]\right\} = W_{\mathrm{L}} \cdot E\left(\Xi_t\right).$$