

# Testing for multiple level shifts in $I(0)$ and $I(1)$ stochastic processes\*

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## Abstract

The paper analyzes the detection and estimation of multiple level shifts regardless of the order of integration of the time series. We show that it is possible to extend the sequential testing procedure of Bai and Perron (1998) to the  $I(1)$  non-stationary case so that a unified framework based on this approach can be applied. The performance of the test statistic is carried out, establishing a comparison with other existing proposals in the literature.

**Keywords:** Unit root, multiple structural breaks

**JEL codes:** C12, C22

## 1 Introduction

The assessment of the order of integration of time series requires the use of test statistics that take into account features that can bias the persistence of the recurrent shocks that affect the time series. One of these features is the existence of structural breaks, i.e., shocks that are not recurrent but that have a large effect on the time series.

Perron (1989) noticed that the inference drawn from the Dickey-Fuller (DF) test statistic can be seriously plagued if the presence of structural breaks is not accounted for. This situation is due to the dependence shown by the limiting distribution of the unit root tests statistics on the type, number and position of the structural breaks – see Perron (1989, 1990), Perron and Vogelsang (1992) and Montañés and Reyes (1998). In this paper, we focus on one particular type of structural breaks, i.e., the ones that only affect the level of the time series. Perron (1990) shows that the limiting distribution of the DF test statistic is invariant to level shifts of fixed magnitude, although not accounting for such level shifts can affect the empirical power of the unit root test statistics. However, the empirical power can also be affected if irrelevant level shifts are specified when computing the unit root tests. Consequently, the key question is how to assess the presence of structural breaks affecting the level of the time series, regardless of its order of integration.

There are different proposals in the literature that address this issue. Perron and Yabu (2009) consider one structural break for trending time series, with three different types of effects – change in the level (Model I), in the slope (Model II) or both (Model III). Saygındoy and Vogelsang (2011) cover the three models in Perron and Yabu (2009), but also consider the

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case of non-trending variables. Kejriwal and Perron (2010) generalize the proposal in Perron and Yabu (2009) to multiple structural breaks, but just focusing on Model II and Model III – i.e., they do not cover the case of structural breaks affecting only the level of the time series. Finally and to the best of our knowledge, Harvey, Leybourne and Taylor (2010) is the only proposal in the literature that deal with the case of multiple level shifts.

In this paper we focus on the issues of testing and estimation of multiple level shifts for non-trending time series, regardless of the order of integration of the stochastic process. Our proposal follows Bai and Perron (1998), where a sequential testing procedure is designed for I(0) stationary processes, but extending their methodology to the I(1) non-stationary framework.

The paper is organized as follows. Section 2 presents the model that allows for the presence of multiple level shift in I(1) non-stationary processes. Section 3 designs the statistic that is used to test the presence of level shifts in a sequential procedure. Section 4 deals with the estimation of the break dates. Section 5 investigates the finite sample performance of the proposed test statistic, and comparing it with other proposals available in the literature. Finally, Section 6 concludes with some remarks. The proofs are collected in the appendix at the end of the paper.

## 2 The model

Let it  $\{y_t\}_{t=1}^T$  be a stochastic process with the data-generating process (DGP) given by:

$$y_t = \sum_{j=1}^m \gamma_j DU_{j,t} + u_t \quad (1)$$

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad (2)$$

with  $\rho = 1$  and  $u_0 = O_p(1)$ . The structural breaks are modeled through the definition of the dummy variables  $DU_{j,t} = 1$  for  $t > \lfloor \lambda_j T \rfloor$ , 0 otherwise, where  $\lfloor \cdot \rfloor$  denotes the integer value of the quantity between parenthesis,  $\lambda_j \in \Lambda(\epsilon)$ ,  $j = 1, \dots, m$ , are the break fractions, and  $\Lambda(\epsilon)$  is a closed subset in  $(0, 1)$  that defines the admissible values of the break fractions. Finally, we assume that  $\varepsilon_t$  is a stochastic process that satisfies the following conditions.

**Assumption:**  $\varepsilon_t = C(L)\eta_t$  with  $\theta(L) = \sum_{j=0}^{\infty} C_j L^j$  with  $C(1)^2 > 0$  and  $\sum_{j=0}^{\infty} j |C_j| < \infty$ , where  $\{\eta_t\}_{t=1}^T$  is an *iid* sequence of with mean zero, variance  $\sigma_\eta^2$  and finite fourth moment. The long-run variance of  $\varepsilon_t$  is given by  $\sigma^2 = \lim_{T \rightarrow \infty} T^{-1} E \left( \sum_{t=1}^T \varepsilon_t \right)^2 = \sigma_\eta^2 C(1)^2$ .

The magnitude of the level shifts are defined as:

$$\gamma_j = \gamma_j^* T^{1/2}, \quad (3)$$

where here  $T^{1/2}$  is the Pitman's drift and  $|\gamma_j^*| < \infty$ ,  $j = 1, \dots, m$ . It is worth mentioning that the definition of the break magnitude using (3) is important to avoid having structural breaks with negligible effects in the limit. This also implies that a consistent estimation of the break dates can be obtained, something that is not possible if the magnitude of the breaks is fixed and the stochastic process is I(1).

## 3 Test statistic

As mentioned above, this paper follows the sequential approach in Bai and Perron (1998) to test the presence of multiple level shifts for I(1) non-stationary processes. Let us consider the rescaled sum of squared residuals (*RSSR*) computed using the vector of break points  $T^0 = (T_1^0, T_2^0, \dots, T_m^0)'$

$$RSSR(T^0) = T^{-2} \left[ \sum_{t=1}^{T_1^0} \hat{y}_t^2 + \dots + \sum_{t=T_{i-1}^0+1}^{T_i^0} \hat{y}_t^2 + \dots + \sum_{t=T_m^0+1}^{T-1} \hat{y}_t^2 \right],$$

where the 0 subscript denotes the true break dates. Further, define the rescaled sum of squared residuals where an additional break is considered in the  $i$ -th segment

$$RSSR(T^0, \tau) = T^{-2} \left[ \sum_{t=1}^{T_1^0} \hat{y}_t^2 + \cdots + \sum_{t=T_{i-1}^0+1}^{\tau} \hat{y}_t^2 + \sum_{t=\tau+1}^{T_i^0} \hat{y}_t^2 + \cdots + \sum_{t=T_m^0+1}^{T-1} \hat{y}_t^2 \right].$$

The null and alternative hypothesis in which we are interested in are

$$\begin{cases} H_0 : & m \text{ structural breaks} \\ H_1 : & m + 1 \text{ structural breaks} \end{cases},$$

so that it is possible to proceed in a sequential fashion testing the null hypothesis of no structural break ( $m = 0$ ) against the alternative hypothesis of one structural break ( $m = 1$ ). If the null hypothesis is rejected, we can proceed testing in a second stage the null hypothesis of  $m = 1$  against the alternative hypothesis of  $m = 2$ , and so on. The sequential testing stops when the null hypothesis cannot be rejected.

Following Bai and Perron (1998) we define the test statistic:

$$F_T(m+1|m) = \hat{\sigma}^{-2} \left[ T^{-2} SSR(T^0) - \min_{1 \leq i \leq m+1} \min_{\tau/T \in \Lambda_i(\epsilon)} T^{-2} SSR(T^0, \tau) \right] \quad (4)$$

$$= \max_{1 \leq i \leq m+1} \max_{\lambda_\tau \in \Lambda_i(\epsilon)} \left[ T^{-2} \hat{\sigma}^{-2} \left[ \sum_{t=T_{i-1}^0+1}^{T_i^0} \hat{y}_t^2 - \sum_{t=T_{i-1}^0+1}^{\tau} \hat{y}_t^2 - \sum_{t=\tau+1}^{T_i^0} \hat{y}_t^2 \right] \right],$$

with  $\Lambda_i(\epsilon) = \{(\lambda_{i-1}, \lambda_\tau, \lambda_i); |\lambda_\tau - \lambda_j| \geq \epsilon \ (j = i-1, i)\}$ ,  $\lambda_\tau = \tau/T$ ,  $\epsilon$  being the trimming,  $\hat{\sigma}^2$  a consistent estimate of the long-run variance of  $\varepsilon_t$ , and where  $\hat{y}_t$  denotes the OLS detrended variable that is obtained from the estimation of (1). The limiting distribution of the  $F_T(m+1|m)$  test statistic given in (4) is given in the following Theorem.

**Theorem 1** *Let  $y_t$  be a stochastic process with the DGP given by (1) and (2). Under the null hypothesis that there are  $m$  structural breaks with  $T^0 = (T_1^0, T_2^0, \dots, T_m^0)'$ , the  $F_T(m+1|m)$  test statistic given in (4) converges as  $T \rightarrow \infty$  to*

$$F_T(m+1|m) \Rightarrow \sup_{1 \leq i \leq m+1} \sup_{\lambda_\tau \in \Lambda_i(\epsilon)} \left[ \int_{\lambda_{i-1}^0}^{\lambda_i^0} \left( W(r) - \frac{1}{\lambda_i^0 - \lambda_{i-1}^0} \int_{\lambda_{i-1}^0}^{\lambda_i^0} W(s) ds \right)^2 dr \right. \\ \left. - \int_{\lambda_{i-1}^0}^{\lambda_\tau} \left( W(r) - \frac{1}{\lambda_\tau - \lambda_{i-1}^0} \int_{\lambda_{i-1}^0}^{\lambda_\tau} W(s) ds \right)^2 dr \right. \\ \left. - \int_{\lambda_\tau}^{\lambda_i^0} \left( W(r) - \frac{1}{\lambda_i^0 - \lambda_\tau} \int_{\lambda_\tau}^{\lambda_i^0} W(s) ds \right)^2 dr \right], \quad (5)$$

where  $\Rightarrow$  denotes weak convergence to the associated measure of probability and  $W(r)$  is a standard Brownian motion.

The proof is given in the appendix. As can be seen, the limiting distribution depends on the number of structural breaks and the position of the structural breaks that is specified under the null hypothesis. In Table 1 we report the asymptotic critical values for the null hypothesis of no

structural break when the trimming is set at  $\epsilon = 0.15$ . These critical values are computed using Monte Carlo simulations with 300 steps to approximate the Brownian motions of the limiting distribution and 50,000 replications.

Harvey et al. (2010) propose an alternative approach based on the generalized fluctuation test statistics. These authors consider the same DGP defined above but distinguishing between two different break magnitude specifications, depending on whether the time series is  $I(0)$  or  $I(1)$ :

$$\gamma_j = \begin{cases} \gamma_j^* T^{-1/2} & |\rho| < 1 \\ \gamma_j^* T^{1/2} & \rho = 1 \end{cases},$$

and propose to use the generalized fluctuation tests:

$$S_0 = \sigma_u^{-1} T^{1/2} \max_{t/T \in \Lambda(\epsilon)} \frac{\sum_{i=1}^{\lfloor \frac{w}{2} T \rfloor} y_{t+i} - \sum_{i=1}^{\lfloor \frac{w}{2} T \rfloor} y_{t-i+1}}{\lfloor \frac{w}{2} T \rfloor} \quad (6)$$

$$S_1 = \sigma^{-1} T^{-1/2} \max_{t/T \in \Lambda(\epsilon)} \frac{\sum_{i=1}^{\lfloor \frac{w}{2} T \rfloor} y_{t+i} - \sum_{i=1}^{\lfloor \frac{w}{2} T \rfloor} y_{t-i+1}}{\lfloor \frac{w}{2} T \rfloor}, \quad (7)$$

where  $S_0$  is the test statistic to be computed when the time series is  $I(0)$  and  $S_1$  when the time series is  $I(1)$ . As can be seen, the  $S_0$  and  $S_1$  are similar, although they differ in two key issues. First, in the computation of the  $S_0$  statistic we require the estimation of the variance of  $u_t$  in (2), whereas when computing the  $S_1$  statistic we need to compute the long-run variance of  $\varepsilon_t$ . Second,  $S_0$  is rescaled by  $T^{1/2}$  whereas  $S_1$  uses  $T^{-1/2}$ . Another key element of the computation of these statistics is the bandwidth  $w$  that is used to compare the averages of the two segments, since the critical values that are required to perform the statistical inference depends on  $w$ . Harvey et al. (2010) show that the larger  $w$  the lower the empirical power of the test statistics, suggesting a value of  $w = 0.10$  for empirical purposes.

## 4 Estimation of the break points

So far, we have assumed that the break points are known a priori, although this situation is rarely found in practice. In general situations, it would be desirable to design a procedure to estimate the break dates in consistent way. In order to do so, we suggest specifying the model in (1) in first difference:

$$\Delta y_t = \sum_{j=1}^m \gamma_j D(T_j)_t + v_t \quad t = 2, \dots, T, \quad (8)$$

where

$$D(T_j)_t = \begin{cases} 1 & t = T_j + 1 \\ 0 & \text{otherwise} \end{cases}; \quad j = 1, \dots, m.$$

Estimate the multiple break points through the minimization of the sum of squared residuals ( $SSR$ ) of the model so that

$$\left( \hat{T}_1, \hat{T}_2, \dots, \hat{T}_m \right) = \arg \min_{(T_1, T_2, \dots, T_m) \in T\Lambda} SSR(T_1, T_2, \dots, T_m),$$

which renders consistent estimates of the break points iff – see Harvey et al. (2010)

$$\gamma_j = \gamma_j^* T^{1/2}.$$

Using the consistent estimates of the break points, we can proceed to compute the  $F_T(m+1|m)$  test statistic and use the critical values that have been computed assuming that the structural

breaks are known. It is worth mentioning that the estimation of the long-run variance  $\sigma^2$  is performed as in Harvey et al. (2010), i.e., we estimate (8) with  $m = m_{\max}$ , where  $m_{\max}$  denotes the maximum number of structural breaks that the whole analysis is going to consider, and then estimate the model by OLS

$$\Delta\hat{v}_t = \pi\hat{v}_{t-1} + \sum_{j=1}^{k-1} \psi_j \Delta\hat{v}_{t-j} + e_t,$$

$t = k + 2, \dots, T$ , and compute  $\hat{\sigma}_e^2 = (T - 2k - 1)^{-1} \sum_{t=k+2}^T \hat{e}_t^2$ , with  $k$  selected so that it satisfies that as  $T \rightarrow \infty$ ,  $1/k + k^3/T \rightarrow 0$  – for instance, the modified information criteria in Ng and Perron (2001) and Perron and Qu (2008) can be used to choose  $k$ . Finally, the long-run variance is obtained as  $\hat{\sigma}^2 = \hat{\sigma}_e^2/\hat{\pi}^2$ . Allowing for the maximum number of structural breaks when computing the long-run variance avoids obtaining a biased long-run variance estimate due to unaccounted structural breaks in the case that there are more structural breaks than the ones considered under the null hypothesis of each step of the sequential testing procedure. This, however, comes at the price of losing power, if more structural breaks than the true number are specified when estimating (8).

## 5 Monte Carlo simulation

In this section we analyze the finite sample performance of the test statistic that has been proposed in this paper and compare it with the test statistic in Harvey et al. (2010). The simulations cover both the cases where the stochastic process is I(0) and I(1). Let us define the DGP:

$$\begin{aligned} y_t &= \gamma DU_t + u_t \\ u_t &= \rho u_{t-1} + \varepsilon_t, \end{aligned}$$

where  $u_0 = 0$ ,  $\varepsilon_t \sim iid N(0, \sigma_\varepsilon^2)$ ,  $\sigma_\varepsilon^2 = 1$  and  $\rho = 1$  for the I(1) case and  $\rho = 0.8$  for the I(0) case. We consider the situation where there is one structural break located in the middle of the sample,  $T_1 = \lfloor 0.5T \rfloor$ , specifying different sample sizes  $T = \{50, 100, 200, 300, 500, 1000\}$ . As for the magnitude of the level shift we define two different set of values according to whether  $y_t$  is I(0) or I(1):

$$\gamma = \begin{cases} \gamma^* & \text{if } y_t \sim I(0) \\ \gamma^* T^{1/2} & \text{if } y_t \sim I(1) \end{cases},$$

with  $\gamma^* = \{0, 1, 5, 10\}$ . As can be seen, when  $y_t \sim I(0)$  the magnitude of the structural break is fixed, where it increases with the sample size when  $y_t \sim I(1)$ . The nominal size is set at the 5% level of significance and 1,000 replications are conducted.

Before addressing the performance of the test statistic, we focus on the estimation of the break date using the procedure discussed above. Figures 1 and 2 present the densities of the estimated break fraction for  $\gamma = 5$  and  $\gamma = 10$ , respectively, each  $T$ . As can be seen, the densities are symmetric and placed around the true value of the break fraction ( $\lambda = 0.5$ ) regardless of the order of integration of the time series. As can be seen, the distribution becomes more concentrated around the true break fraction as the sample size increases. Further, such probability mass concentration increases with the magnitude of the structural break.

Table 2 reports the empirical size of the  $F_T(1|0)$  test statistic when  $y_t \sim I(0)$  – henceforth, BP test – and when  $y_t \sim I(1)$  – henceforth, CG test. As can be seen, both the BP and CG test statistics have the correct size under the respective null hypothesis for all sample sizes that we have considered. When  $y_t \sim I(1)$ , the BP test statistics leads to a clear rejection of the null hypothesis, indicating that the time series is non-stationary. This is not surprising, since in the limit an I(1) process can be interpreted as an stochastic process with infinite structural

changes. When  $y_t \sim I(0)$ , the CG test statistic converges to zero, something to be expected from the rescaling factor that is associated with this test statistic.

Table 3 focuses on the empirical power of the BP and CG test statistics, compared with the  $S_1$  and  $S_0$  test statistics in Harvey et al. (2010) –  $S_1$  refers to the test statistic that detects the presence of level shifts when the time series is  $I(0)$  and  $S_0$  when the time series is  $I(0)$  for different break magnitudes. Let us first focus on the results for  $\gamma^* = 1$ . As can be seen, when  $y_t \sim I(1)$ , the  $S_1$  outperforms the CG statistic in terms of empirical power, although the empirical power of both statistics tends to the nominal size as  $T$  gets large. This is something to be expected, provided that in this setup we consider the magnitude of the structural break to be fixed, i.e., in the limit the structural break is negligible. The empirical power of the  $S_0$  test statistic tends to zero as  $T$  increases, a feature that is used by Harvey et al. (2010) to design a union test – the  $U$  test. Finally, the BP test statistic shows high values for the empirical power due to the size distortions of this statistic when the time series is  $I(1)$ . On the other hand, when  $y_t \sim I(0)$ , the BP statistic outperforms the  $S_0$  and  $U$  tests, whereas the empirical power of both the CG and  $S_1$  tends to zero.

Similar picture is obtained if the magnitude of the structural breaks depends on the sample size. When  $y_t \sim I(1)$ , the  $S_1$  and  $U$  tests outperform the CG statistic, but now the  $S_0$  test statistic does not tend to zero, so that the proposal of the  $U$  test might be compromised – as mentioned above, Harvey et al. (2010) exploited the feature that the  $S_0$  test statistic tends to zero in this case to design the  $U$  statistic. When  $y_t \sim I(0)$ , the empirical power of the BP,  $S_0$  and  $S_1$  statistics tend to one, whereas the empirical power of the CG tends to zero.

Finally, it should be mentioned that increasing the magnitude of the  $\gamma^*$  parameter to either 5 or 10 does not change the qualitative results of the empirical power analysis, showing that the performance of the statistics is driven by whether the magnitude of the structural break is fixed or increasing with  $T$ .

The results that have been obtained in this Monte Carlo simulation reveals that the combined use of the BP and CG test statistics can lead us to better characterize the stochastic properties of time series. If the null hypothesis is rejected by both test statistics, we can conclude that the time series is an  $I(1)$  that has been affected by level shifts. If the null hypothesis is not rejected by either test statistics, we can conclude that the stochastic process is  $I(0)$  with no structural breaks. When the null hypothesis is rejected with the BP test statistic, but not by the CG statistic, then we can conclude that the time series is  $I(1)$  with no structural breaks. This sort of analysis could also be carried out using the HLT test statistics, although the behavior of the  $S_0$  test statistic differs from what the authors used to propose a union test when  $y_t \sim I(1)$  and the magnitude of the structural break is not fixed.

## 6 Conclusions

The paper extends the sequential testing procedure in Bai and Perron (1998) to estimate the number and position of multiple level shifts affecting  $I(1)$  non-stationary processes. Our proposal allows to get a unified framework where the same sequential testing procedure can be used to detect the presence of multiple level shifts regardless of the order of integration of the time series. The limiting distribution of the test statistic that is proposed in this paper is shown to depend on the number and position of the structural breaks that are imposed under the null hypothesis, so specific critical values need to be used in each step of the sequential testing procedure.

The simulation experiment that is conducted compares our proposal with other test statistics available in the literature. We show that in some cases our approach is encompassed by the competing test statistics, although the behaviour of our test statistic is not affected by the assumption made on the magnitude of the structural breaks – which can be either fixed or depending on  $T$  – whereas this is not the case for other existing tests.

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## A Appendix

### A.1 Proof of Theorem 1

Let us first focus on the limit of the expression

$$A(T_{i-1}^0, T_i^0) = T^{-2} \hat{\sigma}^{-2} \left[ \sum_{t=T_{i-1}^0+1}^{T_i^0} \hat{y}_t^2 - \sum_{t=T_{i-1}^0+1}^{\tau} \hat{y}_t^2 - \sum_{t=\tau+1}^{T_i^0} \hat{y}_t^2 \right].$$

The first element of  $A(T_{i-1}^0, T_i^0)$  is given by where

$$T^{-2} \hat{\sigma}^{-2} \sum_{t=T_{i-1}^0+1}^{T_i^0} \hat{y}_t^2 = T^{-2} \hat{\sigma}^{-2} \sum_{t=T_{i-1}^0+1}^{T_i^0} \left( y_t - \frac{1}{T_i^0 - T_{i-1}^0} \sum_{t=T_{i-1}^0+1}^{T_i^0} y_t \right)^2.$$

Provided that  $y_t \sim I(1)$ ,  $T^{-1/2} y_t \Rightarrow \sigma W(r)$  so that by the functional Central Limit Theorem (FCLT)  $(T_i^0/T - T_{i-1}^0/T)^{-1} T^{-3/2} \sum_{t=T_{i-1}^0+1}^{T_i^0} y_t \Rightarrow \sigma (\lambda_i - \lambda_{i-1})^{-1} \int_{\lambda_{i-1}}^{\lambda_i} W(s) ds$ . Consequently, we have

$$T^{-2} \hat{\sigma}^{-2} \sum_{t=T_{i-1}^0+1}^{T_i^0} \hat{y}_t^2 \Rightarrow \int_{\lambda_{i-1}}^{\lambda_i} \left( W(r) - \frac{1}{\lambda_i - \lambda_{i-1}} \int_{\lambda_{i-1}}^{\lambda_i} W(s) ds \right)^2 dr.$$

The same applies to the other two elements of  $A(T_{i-1}^0, T_i^0)$  so that we obtain

$$\begin{aligned} A(T_{i-1}^0, T_i^0) &\Rightarrow \int_{\lambda_{i-1}^0}^{\lambda_i^0} \left( W(r) - \frac{1}{\lambda_i^0 - \lambda_{i-1}^0} \int_{\lambda_{i-1}^0}^{\lambda_i^0} W(s) ds \right)^2 dr \\ &\quad - \int_{\lambda_{i-1}^0}^{\lambda_\tau} \left( W(r) - \frac{1}{\lambda_\tau - \lambda_{i-1}^0} \int_{\lambda_{i-1}^0}^{\lambda_\tau} W(s) ds \right)^2 dr \\ &\quad - \int_{\lambda_\tau}^{\lambda_i^0} \left( W(r) - \frac{1}{\lambda_i^0 - \lambda_\tau} \int_{\lambda_\tau}^{\lambda_i^0} W(s) ds \right)^2 dr. \end{aligned}$$

Finally, by the FCLM theorem, we obtain

$$\begin{aligned} F_T(m+1|m) &\Rightarrow \sup_{1 \leq i \leq m+1} \sup_{\lambda_\tau \in \Lambda_i(\epsilon)} \left[ \int_{\lambda_{i-1}^0}^{\lambda_i^0} \left( W(r) - \frac{1}{\lambda_i^0 - \lambda_{i-1}^0} \int_{\lambda_{i-1}^0}^{\lambda_i^0} W(s) ds \right)^2 dr \right. \\ &\quad - \int_{\lambda_{i-1}^0}^{\lambda_\tau} \left( W(r) - \frac{1}{\lambda_\tau - \lambda_{i-1}^0} \int_{\lambda_{i-1}^0}^{\lambda_\tau} W(s) ds \right)^2 dr \\ &\quad \left. - \int_{\lambda_\tau}^{\lambda_i^0} \left( W(r) - \frac{1}{\lambda_i^0 - \lambda_\tau} \int_{\lambda_\tau}^{\lambda_i^0} W(s) ds \right)^2 dr \right]. \end{aligned}$$



Table 1: Percentiles of the limiting distribution of the  $F(m|m+1)$  test statistic under the null hypothesis of  $m$  structural breaks

$m$	90%	95%	97.5%	99%
0	0.2715	0.3710	0.4763	0.6105

Table 2: Empirical size ( $\gamma^* = 0$ ) of the Bai and Perron (BP) and Carrion-i-Silvestre and Gadea (CG) test statistics

$T$	I(1)		I(0)	
	CG	BP	CG	BP
50	0.033	0.889	0	0.110
100	0.036	0.900	0	0.073
200	0.047	0.913	0	0.052
300	0.036	0.909	0	0.053
500	0.048	0.907	0	0.043
1000	0.040	0.913	0	0.042

Table 3: Empirical power of the Harvey, Leybourne and Taylor (HLT), BP and CG test statistics, with  $w = 0.10$

$\gamma^* = 1$											$\gamma^* = T^{1/2}$									
I(1)						I(0)					I(1)					I(0)				
HLT						HLT					HLT					HLT				
$T$	S1	S0	U	CG	BP	S1	S0	U	CG	BP	S1	S0	U	CG	BP	S1	S0	U	CG	BP
50	0.52	0.29	0.57	0.12	0.87	0.02	0.15	0.16	0	0.82	0.98	0.76	0.99	0.44	0.92	1	1	1	0.64	1
100	0.12	0.04	0.14	0.09	0.87	0	0.06	0.06	0	1	0.97	0.58	0.98	0.41	0.87	1	1	1	0.20	1
200	0.09	0.01	0.10	0.05	0.89	0	0.17	0.17	0	1	0.98	0.45	0.98	0.39	0.92	1	1	1	0.05	1
300	0.08	0.01	0.09	0.03	0.94	0	0.26	0.25	0	1	0.99	0.46	0.99	0.36	0.95	1	1	1	0.01	1
500	0.06	0	0.06	0.04	0.93	0	0.52	0.52	0	1	1	0.38	1	0.32	0.93	1	1	1	0.02	1
1000	0.06	0	0.06	0.05	0.93	0	0.95	0.95	0	1	0.98	0.41	0.98	0.31	0.96	1	1	1	0	1

$\gamma^* = 5$											$\gamma^* = 5T^{1/2}$									
I(1)						I(0)					I(1)					I(0)				
HLT						HLT					HLT					HLT				
$T$	S1	S0	U	CG	BP	S1	S0	U	CG	BP	S1	S0	U	CG	BP	S1	S0	U	CG	BP
50	0.86	0.60	0.87	0.25	0.90	0.92	0.94	0.95	0.20	1	1	1	1	1	1	1	1	1	1	1
100	0.54	0.15	0.55	0.16	0.91	0.75	0.97	0.97	0	1	1	1	1	1	1	1	1	1	1	1
200	0.28	0.03	0.28	0.06	0.92	0.43	0.97	0.97	0	1	1	0.98	1	1	1	1	1	1	1	1
300	0.12	0.00	0.12	0.06	0.89	0.25	0.99	0.99	0	1	1	0.99	1	1	1	1	1	1	1	1
500	0.12	0.01	0.12	0.05	0.88	0.10	0.97	0.97	0	1	1	1	1	1	1	1	1	1	1	1
1000	0.08	0	0.08	0.06	0.95	0.01	0.96	0.96	0	1	1	0.98	1	1	1	1	1	1	1	1

$\gamma^* = 10$											$\gamma^* = 10T^{1/2}$									
I(1)						I(0)					I(1)					I(0)				
HLT						HLT					HLT					HLT				
$T$	S1	S0	U	CG	BP	S1	S0	U	CG	BP	S1	S0	U	CG	BP	S1	S0	U	CG	BP
50	1	0.87	1	0.69	1	1	1	1	0.97	1	1	1	1	1	1	1	1	1	1	1
100	0.97	0.58	0.98	0.41	0.87	1	1	1	0.20	1	1	1	1	1	1	1	1	1	1	1
200	0.81	0.24	0.82	0.19	0.94	1	1	1	0	1	1	1	0.99	1	1	1	1	1	1	1
300	0.53	0.07	0.52	0.19	0.87	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1
500	0.34	0.04	0.35	0.10	0.86	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1
1000	0.16	0.01	0.17	0.07	0.90	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1

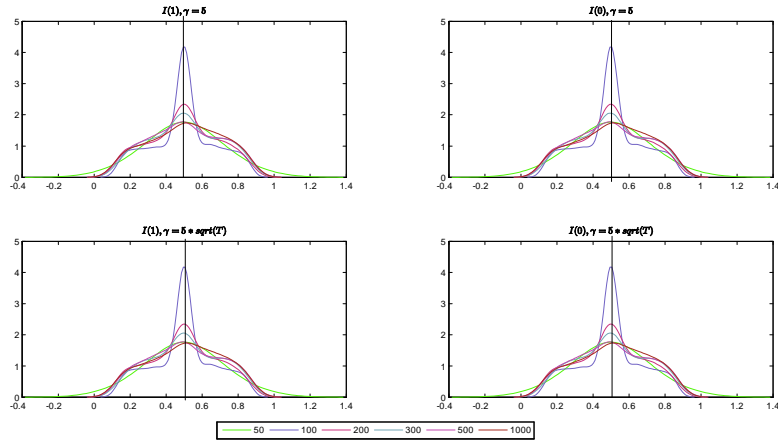


Figure 1: Densities of the break dates estimates,  $\gamma^* = 5$

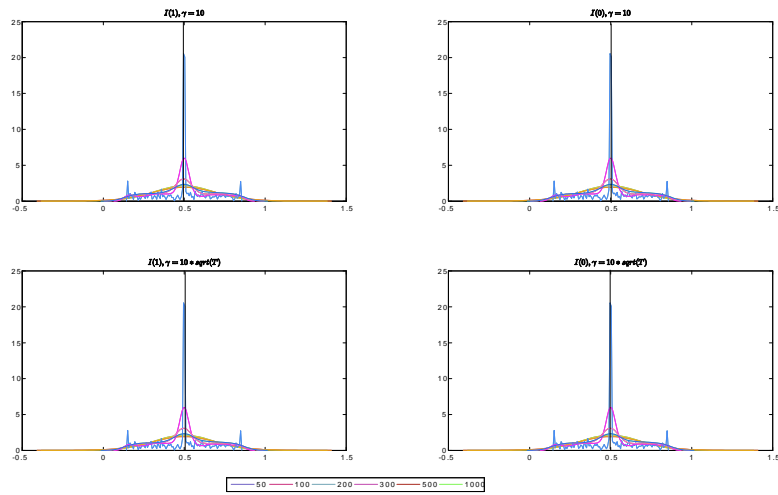


Figure 2: Densities of the break dates estimates,  $\gamma^* = 10$