# On Path–dependency of Constant Proportion Portfolio Insurance strategies

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**Abstract**

This paper evaluates the path–dependency/independency of the most widespread Portfolio Insurance strategies. In particular, we look into various Constant Proportion Portfolio Insurance (CPPI) structures and compare them to the classical Option Based Portfolio Insurance (OBPI) and with naive strategies such as Stop-loss Portfolio Insurance (SLPI).

The paper is based upon conditional Monte Carlo simulations and we show that CPPI strategies with a multiplier higher than 1 are extremely path-dependent and that they can easily get cash-locked, even in scenarios when the underlying at maturity can be worth much more than initially. This likelihood of being cash-locked increases with maturity of the CPPI as well as with properties of the underlying's dynamics and is a major drawback to investors.

To emphasise path dependency of CPPIs, we show that even in scenarios where the investor correctly "guesses" a higher future value for the underlying, CPPIs can get cash-locked and lead to losses. This path-dependency problem is specific of CPPIs, it goes against the European-style nature of most traded CPPIs, and it does not occur in the classical case of OBPI strategies.

We expect that this study will contribute to reinforce the idea that CPPI strategies suffer from a serious design problem.

**Keywords:** Portfolio Insurance, CPPI, OBPI, SLPI, path-dependencies, cash-lock, Conditioned GBM Simulations

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# 1 Introduction

The idea of introducing insurance in investment portfolios was first proposed by Leland and Rubinstein (1976). The main motivation was to prevent the contagious disinvestment movements observed in the stock market crash of 1973-74, which led to the loss of significant potential gains in the subsequent 1975 rise.

Therefore, a portfolio insurance (PI) strategy would consist of an asset allocation strategy between a risk-free asset and a risky asset, so that the combination would give the investor both security, as a percentage of the initial investment guaranteed at maturity, and the possibility of some participation in upside performance of the risky asset.

Leland and Rubinstein (1976) developed the first PI strategy, the Option Based Portfolio Insurance (OBPI), realising that the risky asset can be insured by a put option written on it and whose strike price is the amount to be insured. Although listed options are most of the times not available for long maturities and with adequate prices for most investors, this difficulty could be easily replaced by theoretically modeling the risky asset dynamics. Following the work of Merton (1971), at the time the obvious was to consider the recently developed Black and Scholes (1973) pricing model (BS model). Using the BS model, Rubinstein (1985) proposed an alternative to the static OBPI, based upon the dynamic replication of an option. This synthetic OBPI, is an asset allocation strategy between a risky and a risk-free asset, based upon the delta-heging of options, and is what in fact is used by the industry. In fact, in most of the literature term "synthetic OBPI" is shortened to just "OBPI", as the static OBPI is rarely used.

A few years later, Perold (1986) proposed an alternative PI strategy, the Constant Proportion Portfolio Insurance (CPPI). CPPI strategies were understood as a possible solution of the Merton (1971) problem, for an investor with hyperbolic absolute risk aversion (HARA) utility function. That is, CPPIs were proposed as possible solutions to a very concrete mathematical problem, under extreme assumptions, not only on the risky asset dynamics, but also on the way investors make decisions. For further discussion of this issue see, for instance, Kingston (1989).

No matter the reasons underlying its creation, a CPPI strategy can be understood as a model-free dynamic asset allocation between a risky and a risk-free asset, that is able to guarantee a certain percentage of the initial investment at maturity, just like an OBPI. From a design point of view, however, it is considerably simpler than the classical synthetic OBPI. Its simpler implementation made CPPIs very appealing to a great number issuers, who did not need to rely on any model to manage them. The term "constant proportion" derives from the fact that for every rebalancing date, the amount of the portfolio invested in the risky asset (exposure) is proportional to the so called "cushion". This cushion is nothing but the difference between the total portfolio value at that instant, and the present value of the amount insured at maturity. This present value is known as the "floor". The proportionality factor is fixed at inception for the entire investment period as a "multiplier", $m > 1$.

Since the first appearance of OBPI and CPPI strategies, an extensive literature has sprouted on the subject, with different objectives and methodologies.

Most studies concerned the theoretical properties of continuous-time PI, see for instance Bookstaber and Langsam (1988) or Black and Perold (1992), and references therein. This stream of the literature was mainly focused in solving an optimisation mathematical problem, that arises based upon the classical expected utility theory of Von Neumann and Morgenstern (1944), and assumes investors maximize their end-of-period expected utility. Recently, Balder et al. (2010) has shown that even under the classical assumptions, CPPIs would only be optimal strategies. forcing the exogenously given guarantee into the utility maximisation problem itself. Nowadays, however, the classical expected utility theory itself has been under discussion in academia since the emergence of behavioural finance. Behavioural finance argues that some financial phenomena can plausibly be understood using models in which some agents are not fully rational. For an overview on behavioural finance we refer the reader to survey of Barberis and Thaler (2003). Dichtl and Drobetz (2011) and Gaspar and Silva (2015) evaluate to which extent different behavioural theories would help understanding PI investments. Their results show these theories may indeed help understanding the usage of some (but for all) PI strategies. In particular, CPPI strategies cannot be understood in this context.

A second stream of the literature focused on comparing the strategies with respect to performance, or distribution of returns and stochastic dominance, using Monte Carlo simulations, and relying on theoretical models for the underlying risky asset. Initially the standard approach was to consider a BS model (see e.g. Black and Rouhani (1989), Bertrand and Prigent (2002, 2005) and Bertrand and Prigent (2011)), comparing the strategies mainly in terms of risk and performance measures. Later the analysis focused on distribution properties and stochastic dominance. See, for instance, Annaert et al. (2009) and Zagst and Kraus (2011), Bertrand and Prigent (2011) or Costa and Gaspar (2014). This literature also extended the classical BS model and considered all sorts of alternative models for the underlying risky asset, including jumps-models as in Cont and Tankov (2009) or regime switching models as in Weng (2013). Here the results are mixed, but the more recent studies clearly show that CPPI strategies, present an odd distribution of returns with high probability of returns very close to the floor and quite very low probability of extreme positive returns. Moreover, CPPIs with a multiplier higher than 1 tend to be stochastically dominated by naive portfolio strategies.

A third stream of the literature uses observed empirical densities instead of models, which is considered to be more realistic. Cesari and Cremonini (2003), Köstner (2004) compare an extensive variety of the most used PI strategies – OBPI and CPPI – arriving at the conclusion that CPPI has better performance only in bear and no-trend markets. Almeida and Gaspar (2012) show however, even in that case, CPPIs, are still dominated by naive strategies.

Finally, more recently, a fourth stream has emerged, first identifying problems with the design of CPPI strategies and then proposing modifications to overcome the identified problems. In terms of proposed modifications Pain and Rand (2008) summarised some of the developments until then, from which we highlight the “cushion insurance” of Prigent and Tahar (2005). After that it is worth mention, the dynamics proportions proposed by Chen et al. (2008) or the contingent retracted floor of Lee et al. (2013). Hocquard et al. (2015) proposes alternative strategies with pre-specified distributional properties that present much better results than CPPIs and Bernard and Kwak (2016) suggests modifications taking the perspective of the pension funds industry and long-term investors.

For a more detailed overview on PI, we refer to the survey study and the encyclopedia article of Ho et al. (2010) and Ho et al. (2013), respectively.

This paper is closest to the second stream of the literature, but differs from the existing literature by (i) taking a completely different perspective and, (ii) using, in the context of PI, a recently proposed Monte Carlo *conditional* simulation technique.

We are going to consider the *perspective of an investor* who believes the underlying risky asset will have a positive performance until maturity (if he does not expect that he would not invest in the risky asset to start with). We also consider our investor dislikes risk, so despite his positive expectation, he would still like to guarantee some percentage of the initial investment at maturity (in case his expectations do not realize). We assume nothing else about our investor.

Similarly to most of the literature, we consider a BS model for the underlying risky asset, i.e. we assume it follows a Geometrical Brownian Motion (GBM). Nonetheless, this represent no limitations as our results would only be even stronger if we would consider empirical densities or jump models. See the detailed discussion in Almeida and Gaspar (2012) on what concerns empirical densities, or on the impact of jump models. Our idea here is to take the classical setup and to consider an investor with some subjective expectation about the future value of the risky underlying at maturity, $S_T$ . We then use path dependent Gaussian processes vectorial *simulation technique* proposed in Sousa et al. (2015) to impose a terminal value on $S_T$. This allows us to focus on path-dependencies/independencies of the various strategies. We look into various CPPIs and compare them not only with the classical OBPI but also with naive portfolio insurance strategies, in the spirit of Costa and Gaspar (2014).

We consider CPPIs with multipliers, $m = 1, 3, 5$. Multipliers of the order of 3, 5 or even higher are quite common in the real life products. On the other hand, a CPPI 1 ($m = 1$) is not a true CPPI. In fact, a CPPI 1 is simple static naive strategy where one deposits at inception the present value of the future guarantee and invests only the remaining in the risky-asset. In this paper CPPI 1 is, thus, a naive portfolio insurance strategies one should always keep in mind as feasible and that requires no management whatsoever. A second naive strategy is, of course, the well-known Stop-loss Portfolio Insurance (SLPI), that consist in investing at inception $100\%$ in the risky asset, keep track of its evolution and disinvest from the risky asset to a risk-free deposit as soon as its value touches the present value of the future guarantee. This is always feasible and portfolio insurance strategies that did emerge after the crisis of 1973-74 intended to improve on possible negative effects of massive disinvestments in crisis periods, associated with this strategy. Consistently with Costa and Gaspar (2014) we will show that under some circumstances CPPIs with $m > 1$ also lead to massive disinvestments, as they are also extremely path-dependent with the disadvantage that their maturity-return distribution is worse than the SLPI return distribution, for risk-averse investors.

Our approach allows us to clearly evaluate the path–dependency/independency of the various PI strategies. For the particular case of CPPIs we show: (i) they are extremely path-dependent, and that (ii) they can easily get cash-locked, even in scenarios when the underlying at maturity can be worth much more than initially. We expect that this study will contribute to reinforce the idea that CPPIs are ill-designed PI strategies, bearing therefore an enormous design risk to investors.

The paper is organised as follows. In the following section, we expose the setup and notation necessary to comprehend the construction of the different strategies under analysis. We also provide a final example depicting a situation where the CPPI 3 and 5 could become cash-locked. In Section 3 we present the methodology used in the simulation of the risky asset, the parameter scenarios involved and and the statistical methods. Section 4 presents and discussed the results obtained. Section 5 summarises our main conclusions and discusses further research on this topic.

## 2 Portfolio Insurance

We consider the price process of a risky asset , $S$ (e.g. stock), whose dynamics are driven by a geometrical Brownian motion (GBM) , and a risk-free asset, $B$.

A PI strategy can be represented for every $t \in [0, T]$ by the pair $(\nu^B, \nu^S)$, which denotes the exposure to the risky and risk-free assets, respectively. The strategy's value $(V_t^p)_{t\in[0,T]}$ is hence given by $V_t^p = \nu_t^B B_t + \nu_t^S S_t$ (Balder and Mahayni, 2009). We only consider self-financing PI, i.e. with no exogenous injection or withdrawal of money during $]0, T[$. Naturally, the CPPI, OBPI and SLPI strategies correspond to this type of strategies which can only purchase more assets if they have previously sold others. This self-financing property implies

$$dV_t^p = \nu_t^B dB_t + \nu_t^S dS_t, \tag{1}$$

and typically the insured component of the investment can be translated into the expression

$$B_T = \eta V_0^p, \tag{2}$$

where $\eta$ (typically ranging from 80% to 100%) is the percentage of the initial invested capital to be insured. Assuming non-arbitrage, at $t = 0$ we have $\nu_0^S > 0$ and hence $V_0^p > \nu_0^B B_0$, which mean $\eta$ is limited to the future value of the initial portfolio investment, and therefore $0 \leq \eta < e^{rT}$ (Zagst and Kraus, 2011). In other words, an investor can never insure more than the capitalised value of its investment.

### 2.1 Stop-Loss Portfolio Insurance (SLPI)

As its name suggests, this simple strategy consists on the portfolio being entirely invested into the risky asset, and if it falls below the investor's pre-established floor $F_t$, the portfolio is automatically rebalanced into the risk-free asset. The floor is a representation of the bond with continuously compound deterministic interest $r$ in $[t, T]$:

$$F_t = F_T e^{-r(T-t)}. \tag{3}$$

Thus, the portfolio value of the SLPI strategy can be formally defined by

$$V_t^{SLPI} = \frac{V_0}{S_0} S_t 1_{\{\tau > t\}} + F_t 1_{\{\tau \leq t\}}, \tag{4}$$

where $\tau = \inf\{t > 0 : V_t^{SLPI} = F_t\}$ is the first instant that the portfolio 'touches' the floor barrier, if it exists.[1] The indicator functions $(1_{\{\tau>t\}}, 1_{\{\tau\leq t\}})$ are respectively $(1, 0)$ if $\tau \notin ]0, t]$ - i.e., in this period the portfolio never touched the barrier - and $(0, 1)$, otherwise. Therefore, SLPI is clearly a path-dependent strategy because its value at $t$ depends on wether the risky asset path dropped at touched the floor before $t$, or not. In other words, if we imagine two stock paths leading to the same ending, if one has reached the floor barrier and the other has not, the final portfolio value will be $F_T$ and $S_T$, respectively.

[1] Note that we do not include $t = 0$ because it would mean deliberate investment in the risk-free asset. Also note that if $\{t > 0 : V_t^{SLPI} = F_t\} = \emptyset$, then its infimum is $\infty$ and the definition also holds.

## 2.2 Option Based Portfolio Insurance (OBPI)

An European option is a $T$ – contingent claim where the investor purchases the right, but not the obligation, to buy (call) or sell (put) the underlying asset at maturity at a specified strike price $K$.

$$\Phi(S_T) = \max[(S_T - K), 0]. \tag{5}$$

An OBPI strategy, as introduced by Leland and Rubinstein (1976) can be understood as a static strategy that consists of investing the amount $F_0 = F_T e^{-rT}$ in the risk-free asset and using the remaining to buy $q$ European call options with strike price $K$, such that:

$$V_0^{OBPI} = qCall(0, S_0) + F_T e^{-rT}, \tag{6a}$$

$$F_T = qK, \tag{6b}$$

where $F_T = \eta V_0^{OBPI}$ - recall Equation (2) - and thus, both $q$ and $K$ are uniquely determined. Since this OBPI is a static PI strategy, once we have all parameters set to begin the strategy, no further calculations are required until $t = T$ where the exercise of the contract takes place and the value of the portfolio is given by

$$V_T^{OBPI} = q\max[(S_T - K), 0] + F_T = \begin{cases} qS_T & \text{if } S_T > K \\ qK & \text{if } S_T < K \end{cases}. \tag{7}$$

As it was mentioned before, listed options are usually unreachable to most investors. An alternative solution to this problem is to dynamically replicate the European call option, creating a dynamic PI strategy consisting of investments in the risky-asset and the risk-free asset only. In other words, we want $(\nu^B, \nu^S)$ such that it matches the performance of an OBPI strategy for every $t \in [0, T[$. Again, since we consider only the simple case of European call option, the model can deliver a closed-form price function for this type of option.

In the B-S framework asset $B$ is also considered a continuously compound zero-coupon bond (ZCB) at risk-free interest rate $r$ and $S_t$ follows a geometric Brownian motion (GBM), i.e.

$$dB_t = rB_t dt \qquad \Rightarrow \qquad B_t = B_T e^{-r(T-t)}, \tag{8a}$$

$$dS_t = S_t(\mu dt + \sigma dW_t) \qquad \Rightarrow \qquad S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \tag{8b}$$

where $(W_t)_{t\in[0,T]}$ is a Brownian motion and $\mu > r \geq 0$ and $\sigma > 0$ are commonly referred to as the *drift* and *volatility parameters*, respectively. With this model, the call function can be obtained by introducing the contract function of Equation (5) as a boundary condition to the B-S partial differential equation, yielding the following solution (for derivation of the B-S equation see, e.g., Hull (2009)):

$$Call(t, S_t) = S_t\,\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2), \tag{9}$$

where $\mathcal{N}(.) \equiv \mathcal{N}(0, 1; .)$ is the cumulative distribution function for the standard normal distribution and

$$d_1 \equiv d_1(t, S_t) = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \quad (10a)$$

$$d_2 \equiv d_2(t, S_t) = d_1 - \sigma\sqrt{T-t}. \quad (10b)$$

Hence, the replicated portfolio is given in terms of asset numbers by $\nu_t^B = q[1 - \mathcal{N}(d_2)]$ and $\nu_t^S = q\mathcal{N}(d_1)$ for $t \in [0, T[$ and for $t = T$, Equation (7) also holds.Note that this strategy is model dependent, as it is necessary to estimate a proper value for $\sigma$, the volatility of the risky asset. Figure 1 illustrates a synthetic OBPI path.

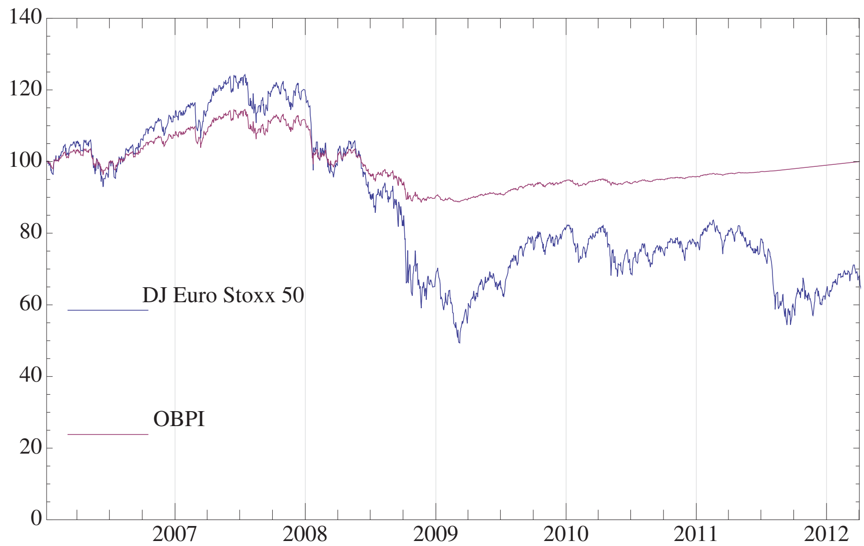



Figure 1: Synthetic OBPI strategy applied over DJ Euro Stoxx 50 index: $V_0 = 100$; $r = 4\%$; $\eta = 100\%$, $\sigma = 23.67\%$; $T = 4.378$ (years from 6/1/2006 to 2/4/2012).

## 2.3 Constant Proportion Portfolio Insurance (CPPI)

### 2.3.1 Standard CPPIs, m >1

A CPPI strategy is a dynamic asset allocation, that at any moment $t$, keeps track of the present value $F_t$ of the future guarantee required by the investor. The process $(F_t)_{t\in(0,T)}$ is called the *floor*. The difference between the portfolio value $V_t^{cppi}$ and the floor, is called *cushion* and defined as $C_t = V_t^{cppi} - F_t$ for all $t \in (0, T)$.

In a CPPI one invests a leveraged amount $m \times C_t$ in the risky asset, and the remaining is the risk-free asset. The *multiplier* $m$ is kept fixed trough the entire investment time and defined in the product's term sheet. Typical multiplier values range from $m = 2$ to $m = 7$. The amount invested in the risky asset is called the *exposure* of the portfolio $V^{cppi}$ to the risky asset and we shall denote it by $(E_t)_{t\in(0,T)}$. The amount invested in the risk-free asset $B$ is simply the rest of the portfolio value $V_t^{cppi} - E_t$, for all $t \in (0, T)$. To compare with Equation (1) one can also write $V = \frac{E_t}{S_t} S_t + \frac{V_t - E_t}{B_t} B_t$, so that $\nu_t^S = \frac{E_t}{S_t}$ and $\nu_t^B = \frac{V_t - E_t}{B_t}$. At any moment, $t \in (0, T)$, the exposure to the risky asset is given by

$$E_t = mC_t = m(V_t^{cppi} - F_t) . \tag{11}$$

Note that if the portfolio value approaches the floor, the cushion diminishes, and so does the exposure to the risky asset. In case the portfolio value touches the floor the cushion is zero and, thus, zero we get exposure to the risky asset in any future date. When that happens the CPPI strategy gets *cash-locked* and will only pay at maturity the guaranteed amount. From the moment it touches the floor a CPPI becomes a static strategy with 100% investment is the risk-free asset (zero-coupon bonds (ZCBs)).

Just for illustration purposes, let us suppose at $t = 0$ an investor invests 100 in a CPPI strategy with $m = 3$ and with full guarantee ($\eta = 100\%$), i.e $F_T = 100$ . Consider $r = 4\%$ and $T = 5$ years, so we have $F_0 = 100e^{-0.04\times 5} \approx 81.87$ and $C_0 = 100 - 81.87 = 18.13$. Therefore, initially we have $E_0 = 3C_0 = 54.39$ invested in the risky asset and the rest $100 - E_0 = 45.61$ in ZCB. After the initial portfolio composition, the exact evolution of $S_t$ plays an important role. Assume, to simplify, that the next trading day takes place exactly a year after ($t = 1$) and $S$ has risen 10%. We have $F_1 = 85.21$ and hence $V_1 = 54.39(1 + 10\%) + 45.61\frac{F_1}{F_0} = 107.3$, because B evolves at the same rate of $F_t, r$. Therefore, $C_1 = 107.3 - 85.21 = 22.09$, $E_1 = 66.27$ and $V_1 - E_1 = 41.03$. An example of CPPI 3 applied to the DJ Euro Stoxx index index is shown at three different dates in Figure 2. We can see in this case that the exposure proportion becomes very short which mean the investment ended practically cash-locked.

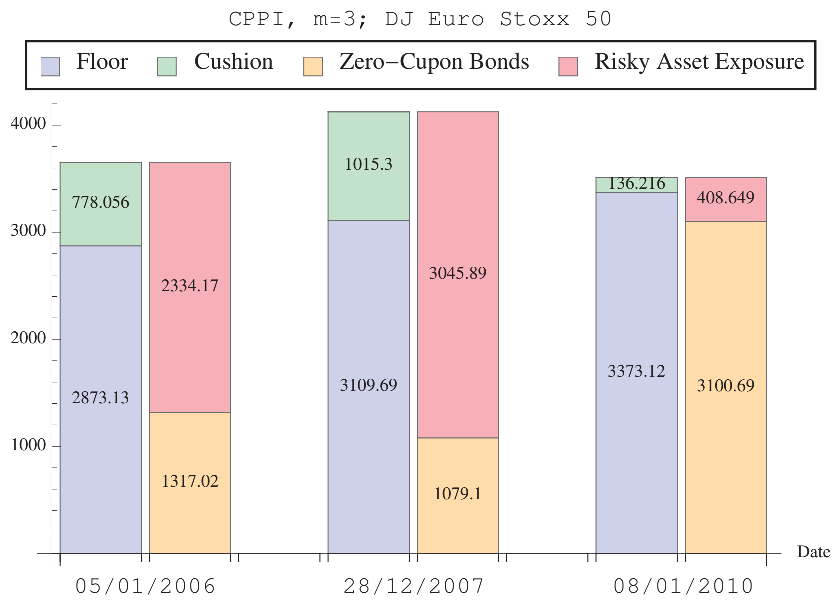



Figure 2: Bar chart of CPPI 3 structure at three different dates. Underlying Asset: DJ Euro Stoxx 50 index.

In the illustration of Figure 2 we considered only three rebalancing dates. In real life situations the rebalancing of CPPI strategies is done either daily or at least weekly, as one needs to prevent for the possibility of crossing the floor within rebalancing dates. Also, most times there is an additional rule establishing a positive lower limit to cushion value (and not zero) as the trigger to invest all in the risk-free asset.

### 2.3.2 The special case of a CPPI 1, $\mathbf{m = 1}$

CPPI strategies with $m = 1$ do not exist in real life. In this study we consider the case of CPPI 1 strategies, as they are simple naive static portfolio insurance strategies. In fact, a CPPI 1 is simple static naive strategy where one invests, at inception, the present value of the future guarantee in the risk-free asset and invests only the remaining in the risky-asset. CPPI 1 is, thus, a naive portfolio insurance strategy one should always keep in mind as feasible and that requires no management whatsoever.

In this study we consider two naive portfolio insurance (PI) strategies – SLPI and CPPI – and three non-naive ones – OBPI, CPPI 3 and CPPI 5. Section 3 describes the methodology used and its financial intuition.

# 3 Methodology

In this paper we focus on the differences across strategies, when both the initial value $S_0$ and the terminal value $S_T$ of the risky asset are known. This analysis is suggested as a way to analyze path-dependencies of PI stratgies. We model stock trajectories using a GBM, however in terms of the simulations we need to impose that all geometric Brownian motion paths are tied to $S_T$ at maturity.

## 3.1 Conditional GBM simulation

To generate the conditioned GBM paths, we use gaussian processes for machine learning regression (GPR), which is given by Rasmussen and Williams (2005). Following this work, applications to different stochastic processes are provided by Sousa et al. (2015), in particular for the GBM. The GBM follows a lognormal distribution, which means its logarithm is a gaussian process. Therefore, we generate a process $y_t$ which is a Brownian Motion with drift and conditioned to $y_n$ (Brownian Bridge) and obtain the GBM by exponentiation i.e. $S_t = S_0 e^{y_t}$.

In the general case, the purpose of GPR is to obtain the non-linear regression function $y = f(\vec{x})$ that maps the data $(\mathbf{X}, \vec{y})$ called the training set, assuming a specific prior gaussian process, i.e $\mathcal{GP} \sim \mathcal{N}(m(\vec{x}, cov(\vec{x}_1, \vec{x}_2))$. The matrix $\mathbf{X}$ gathers the $n$ vectors $\vec{x}_i = x_i^1, \ldots, x_i^d$ which contain the $d$ parameters that originate the corresponding $n$ observations $y_i = f(\vec{x}_i)$ with $i = 0, \ldots, n$. In the present case however, this setting is much more simplified because $\vec{x} = t$ and the training set reduces to the single observation $(t_n = T, y_n = \log \frac{S_T}{S_0})$. The remaining time steps $t_0, t_1, \ldots, t_{n-1}$ are collected in the vector $\mathbf{t}^*$ called the test set and represent the instants where $y_i^* = f(t_i^*)$, $i = 0, \ldots, n-1$ was not observed.[2] The regression process is also gaussian and it is obtained by the mean and covariance functions of the process defined by all the trajectories of the

[2] The arrow representation was only used to represent the general case of a vector with $i = 1, \ldots, d$ different parameters. In this case we use only one parameter, $t$. Different points $k = 0, \ldots, n$ are collected in vectors represented by bold font, e.g. $\mathbf{t} = (t_0, t_1, \ldots, t_n)$

prior process that passes through the training set. Since the process is gaussian, we have

$$\begin{bmatrix} y_n \\ \mathbf{y}^* \end{bmatrix} \sim \mathcal{N}\left( \begin{bmatrix} m(T) \\ \mathbf{m}^* \end{bmatrix}, \begin{bmatrix} cov(T,T) & \mathbf{cov}^{*\top} \\ \mathbf{cov}^* & \mathbf{cov}^{**} \end{bmatrix} \right) \tag{12}$$

where $\mathbf{m}^* = \big(m(0), m(t_1), \ldots, m(t_{n-1})\big)$, $\mathbf{cov}^* = \big(cov(0,T), cov(t_1,T), \ldots, cov(t_{n-1},T)\big)$ and the matrix elements $(cov^{**})_{ij} = cov(t_i^*, t_j^*)$, with $i,j = 0,1,\ldots,n-1$. The conditional distribution is given by

$$p(\mathbf{y}^* \mid \mathbf{t}^*, T, y_n) \sim \mathcal{N}\left( \mathbf{m}^* + \frac{y_n - m(T)}{cov(T,T)} \mathbf{cov}^*, \mathbf{cov}^{**} - \frac{1}{cov(T,T)} \mathbf{cov}^* \mathbf{cov}^{*\top} \right), \tag{13}$$

where one should note that $\mathbf{cov}^*\mathbf{cov}^{*\top}$ must be read as an outer product resulting in a $n \times n$ matrix with elements $cov(t_i,T) \cdot cov(t_j,T)$, $i,j = 0,\ldots,n-1$. The mean and covariance of this process are used to build respectively the regression function and regression confidence, by extending to the whole $\mathbf{t}$ set. Therefore, the posterior process on the data has the following mean and covariance functions

$$m_{\mathcal{D}}(t) = m(t) + \frac{1}{cov(T,T)} cov(t,T)(y_n - m(T)), \tag{14a}$$

$$cov_{\mathcal{D}}(s,t) = cov(s,t) - \frac{1}{cov(T,T)} cov(s,T) cov(t,T), \tag{14b}$$

Hence, using Equations (14a–14b) we can simulate any path of a gaussian process with mean $m$ and covariance $cov$ that passes through (in this case end at) $(T, y_n)$. In our particular framework, we deal with a Brownian motion with mean and covariance given by

$$m(t) = \left( \mu - \frac{\sigma^2}{2} \right) t, \tag{15a}$$

$$cov(s,t) = \sigma^2 \min(s,t), \tag{15b}$$

where $\mu$ and $\sigma$ are again the drift and volatility, respectively. The imposition of the training set $(T, \log \frac{S_T}{S_0})$ will particularize equations 14. Noting that $cov(t_i,T) = \sigma^2 t_i$, $\forall_{i=0,\ldots,n}$, eq. 14a simplifies significantly to

$$m_{\mathcal{D}}(t) = \left( \mu - \frac{\sigma^2}{2} \right) t + \frac{\sigma^2 t}{\sigma^2 T} \left[ \log \frac{S_T}{S_0} - \left( \mu - \frac{\sigma^2}{2} \right) T \right] = \frac{t}{T} \log \frac{S_T}{S_0}. \tag{16}$$

This result as the important meaning that $m_{\mathcal{D}}(t)$ does not depend on $\mu$, which also means that the GBM tied to one point gives place to the natural reparametrization of $\mu$ by $\tilde{\mu} = \frac{1}{T} \log \frac{S_T}{S_0} = \log \left[ \left( \frac{S_T}{S_0} \right)^{1/T} \right]$. We should in fact expect this result, because the risky asset is assumed log-normal which means that the annualized return will be $e^{\tilde{\mu}} = \left( \frac{S_T}{S_0} \right)^{1/T}$. Additionally Equation (14b) can also be reduced to

$$cov_{\mathcal{D}}(s,t) = \sigma^2 (\min(s,t) - st) = \sigma^2 \begin{cases} s - st & , s \le t \\ t - st & , s > t \,. \end{cases} \tag{17}$$

Now the Brownian bridge path can be obtained by (see Glasserman (2003))

$$\mathbf{B} = \mathbf{m}_{\mathcal{D}} + \mathbf{C}\mathbf{Z}, \tag{18}$$

where $\mathbf{C}$ is the Cholesky decomposition of $\mathbf{cov}_{\mathcal{D}}$, i.e. the covariance matrix whose elements are given by Equation (17), and $\mathbf{Z}$ is a vector of the standard normally distributed $\mathcal{N}(0, I)$ random numbers. Finally, the GBM paths are simulated by exponentiation of the Brownian bridge process, i.e $S_t = S_0 e^{B_t}$ so that $S_T = S_0 e^{\log S_T / S_0}$.

Figure 14 illustrates the method described above, presenting two conditional GBM paths for two pre-fixed values at maturity.


Figure 3: Two geometric Brownian motion paths conditioned to $S_T = 100$ (red) and $S_T = 300$ (blue), simulated with gaussian processes for machine learning regression. $T = 5$ and $\Delta t = 1/100$.

XXX TO DO: financial intuition XXX

XXX TO DO: Explain discretization of other strategies or make reference XXX

### 3.2 On CPPI discrete implementation

In the context of a continuous-time model such as the BS model (in Equation (8)), the pair $(\nu^S, \nu^B)$ in Equation (1), for the CPPI is given by

$$dV_t = \frac{E_t}{S_t} dS_t + \frac{V_t - E_t}{B_t} dB_t \,. \tag{19}$$

For the BS model, the CPPI value evolution follows immediately,

$$V_t^{cppi} = V_0^{cppi} \left[ \eta e^{-r(T-t)} + (1 - \eta e^{-rT}) e^{\lambda t} \left( \frac{S_t}{S_0} \right)^m \right], \tag{20}$$

where $\lambda = (1-m)(r + m\frac{\sigma^2}{2})$ and $\eta$ as defined in Equation (2).

We now proceed with an intuitive approach to CPPI in a discrete-time basis, making it more identifiable with the real world.[3]. Contrary to the continuous case, in the “real world”, traders are restricted to the official rebalancing dates defined in the product’s term sheet. Therefore, one must be prudent when choosing the multiplier, as the strategy can only insure that $V_t \geq F_t$ for a limited drop in the market between two consecutive rebalancing dates. The risk of the stock dropping at a rate greater than the threshold is called *gap risk*. The smaller the period between rebalancing dates, the smaller the gap risk. In practice, as previously mentioned, gap risk is avoided by a covenant in the term sheet that allows to transfer the entire investment to the risk-free asset even with a positive (but small) cushion.

Let us consider the simplest case of a partition of the time interval $[0, T]$ consisting of $n+1$ equidistant $t_k$ time steps, i.e $t_0 = 0 < t_1 < \ldots < t_n = T$ and $t_{k+1} - t_k = T/n \equiv \Delta t, \forall_{k=0,\ldots,n}$. Now, from the self financing condition, the discrete form of Equation (19) can be rewritten in terms of $t_k$, i.e $\Delta V_{k+1} \equiv V_{k+1} - V_k$ and is given by (to ease the notation let $x_{t_k} \equiv x_k$):

$$\Delta V_{k+1} = \frac{E_k}{S_k} \Delta S_{k+1} + \frac{V_k - E_k}{B_k} \Delta B_{k+1}. \tag{21}$$

As we consider the non-existence of short-selling, the CPPI exposure has to be defined with an inferior barrier of zero, i.e

$$E_k = \max[mC_k, 0] = \begin{cases} m(V_k^{cppi} - F_k) & \text{if } V_k^{cppi} \geq F_k \\ 0 & \text{if } V_k^{cppi} < F_k. \end{cases} \tag{22}$$

Note that in the continuous case, the null branch is not necessary because continuous rebalancing makes sure that $V_t \geq F_t, \forall t \in [0, T]$. Hence we see that the assets’ weights are $\nu_k^S = \frac{\max[mC_k, 0]}{S_k}$ and $\nu_k^B = \frac{V_k - \max[mC_k, 0]}{B_k}$, so the portfolio value is given by

$$V_{k+1}^{cppi} = \begin{cases} m\left(V_k^{cppi} - F_k\right) \dfrac{S_{k+1}}{S_k} + \left(V_k^{cppi}(1-m) + mF_k\right) \dfrac{B_{k+1}}{B_k} & \text{if } V_k^{cppi} \geq F_k \\ V_k^{cppi} \dfrac{B_{k+1}}{B_k} & \text{if } V_k^{cppi} < F_k \ . \end{cases} \tag{23}$$

Thus, given the inputs $V_0, \eta, r, T$ and $m$ we obtain $F_0$ and $E_0$. For all $k$, with $\frac{B_k}{B_{k-1}} = \frac{e^{-r(T-t_k)}}{e^{-r(T-t_{k-1})}} = e^{r(t_k - t_{k-1})} = e^{rT/n}$ we get $F_k = \frac{B_k}{B_{k-1}} F_{k-1}$, and at last, given $S_k$ we have all that is necessary to know the following $V_k$, ($k = 1, \ldots, n$) by the recursion expression in Equation (23). In other words, in every time step $t_{k+1}$, CPPI algorithm invests the previous $V_k$, allocates $E_k = m(V_k - F_k) \geq 0$ in $S$ and $V_k - E_k$ in $B$, and obtains $V_{k+1}$ by the stochastic variations of $S$ and the known growth of $B$. Figure 4 shows an application of the CPPI strategies on a world stock index, considering as rebalancing dates all trading dates and full capital guarantee at maturity. For this particular real life instance we observe a bad performance of the underlying risky asset, and that all CPPIs (by definition) where able to guarantee the capital. Nonetheless we note cash-lock occurrences for CPPI 3 and 5, but not for CPP 1 that under the circumstances was the best strategy (despite its “naivety”).

[3]For a more formal approach and details we refere to Brandl (2009).

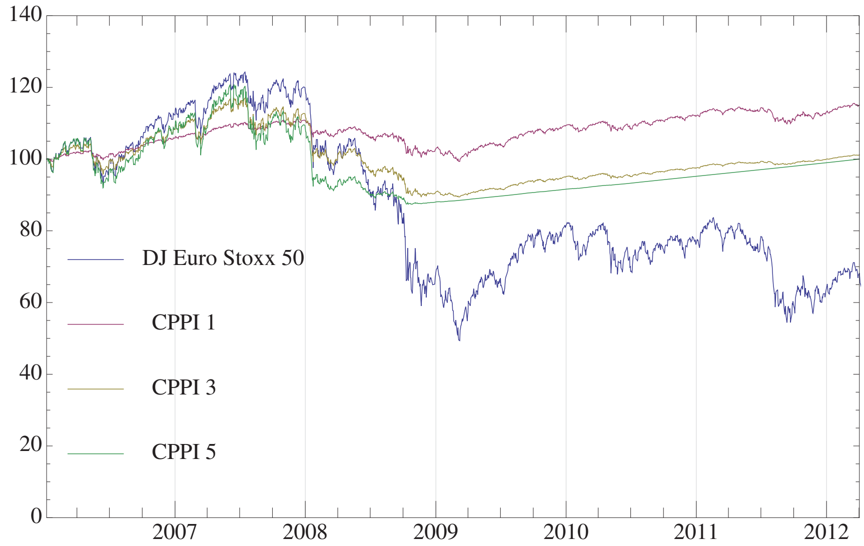



Figure 4: CPPI 1, 3 and 5 applied over DJ Euro Stoxx 50 index. $V_0 = 100$; $r = 4\%$; $\eta = 100\%$.

### 3.3 Parameters

The simulations count on two types of parameters to implement: (i) the procedure parameters, which are fixed for every simulation; and (ii) the scenario parameters, which will assume different values that to recreate different scenarios.

**Procedure parameters:**

- The initial portfolio investment $V_0 = 100$;
- the rebalancing frequency, i.e, constant time increments are $\Delta t = 1/100$, which can be thought as the distance between rebalancing dates measured in years;
- the number of time steps is $n = T/\Delta t$;
- the number of paths / simulations $N = 10000$ (as in Annaert et al. (2009)); and
- the risk-free interest rate $r = 4\%$.The choice of the $4\%$ is among the values generally used in the literature. See, e.g. $5\%$ in Costa and Gaspar (2014), $3\%$ and $4\%$ in Cont and Tankov (2009).

**Scenario parameters:**

- the volatility of the stock, $\sigma$:$\{15\%, 40\%\}$;
- the percentage of the initial portfolio to be insured $\eta$:$\{100\%, 80\%\}$;
- the maturity of the investment $T$:$\{5, 15\}$ ; and
- tsock value at maturity $S_T$:$\{100, 150, 200, 250, 300\}$.

The combination of all scenario parameters result in considering $40$ different scenarios. The main difference on this scenario setup with respect to other literature, is the fixation of $S_T$ (instead to the usual $\mu$ on the risky asset dynamics).

The present work also extends the scenarios used in Costa and Gaspar (2014) by introducing $T = 15$ to the analysis, since long maturities can be found in some PI products. Another particularity is the choice of $S_T$ values all above $S_0 = 100$. The reason is due to the fact that for negative rates of return of the underlying risky asset, PI strategies will return only the guarantee as they end invested almost entirely on the risk-free asset. Here we are concerned mainly concerned with the performance of PI strategies, under scenarios where the underlying risky asset actually performs well until the maturity. These are also the only scenarios where CPPI strategies, with a multiplier higher than one, may outperform the remaining strategies.

Figure 5 below is an illustration of 8 conditioned GBM evolutions with the associated CPPI 1, CPPI 3 and CPPI 5 outcomes. In Figure 5 (a)-(b) we impose $S_T = 100$, in Figure 5 (c)-(d) we impose $S_T = 200$, in in Figure 5 (e) we impose $S_T = 300$, in Figure 5 (f)-(g) we impose $S_T = 300$ and in Figure 5 (h) we impose $S_T = 800$.

A few comments are needed at this point. Although these are just 8 particular paths, it is interesting to notice that in all presented instances the CPPI 5 ends up getting cash-locked, even in the case when the risky asset increases 8 times during the investment period. The same happens to CPP 3 in six out of our eight instances. Moreover, in the instances CPPI 3 did not get cash-locked its outcome is extremely close to the floor.

From a different perspective we realise the investment horizon matters enormously. As the investment horizon increases, the higher is the risk that at some point a CPPI value with a multiplier higher than one, will approach the barrier and eventually get cash-locked. In particular, from Figure 5 (h) we see that even if the risky asset increases drastically early in the investment time, that does not make CPPI 5 ou CPPI 3 less risky, on the contrary, the amplifying effect of the multiplier do allow to potential huge gains in the very beginning, but it also amplifying the speed at which the strategy approaches the floor in an event of a drop is the value of the underlying risky asset.

Finally, we also notice that in all presented instances CPPI 1 – a very naive strategy – outperformed CPPI 3 and CPPI 5.

In the next section, we will of course, focus on more realistic scenarios and for each scenario we look into 10 000 paths. Still, we find the images in Figure 5 illustrative of what we think are the main risks of CPPIs with a multiplier higher than one.

### 3.4 Distribution Analysis and Stochastic Dominance

In order to analyse and confront the aforementioned PI strategies we have chosen two of the most significant statistical methods used in the literature.

One is the direct study of the various PI payoff distributions at maturity. The first four moments are often used in literature because they can easily be interpreted and much information can be withdrawn about the shape of the payoff distributions (see e.g. Prigent and Bertrand (2003); Pezier and Scheller (2011) and Khuman et al. (2008) uses log-moments). Here we opt to present the actual distributions and to compute additional measures of performance, besides the first four moments.

Additionally, as we are specially concerned with the investor's perspective, we also perform a Stochastic Dominance (SD) analysis. Stochastic Dominance was introduced first by Quirk and

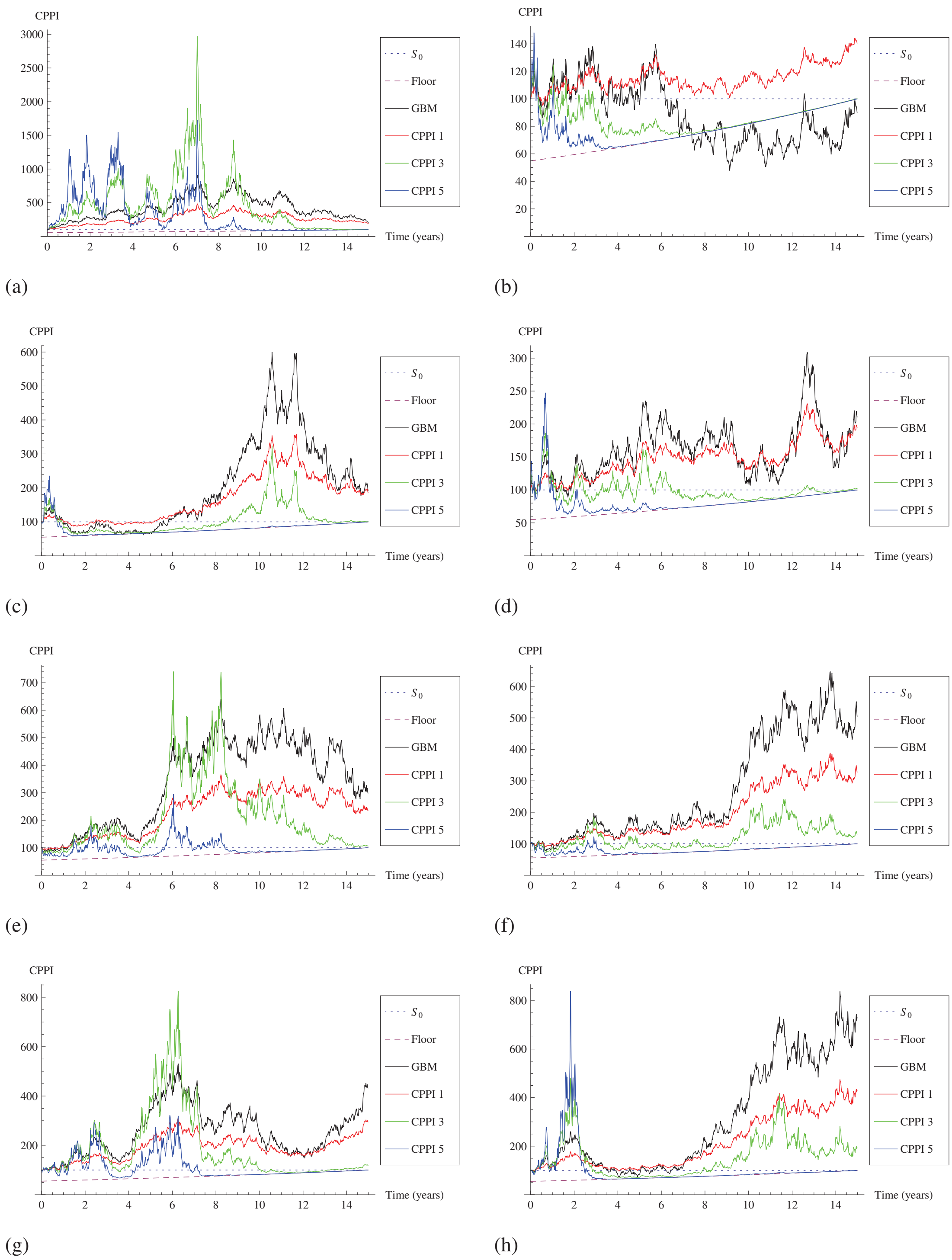



Figure 5: Illustration of GBM conditional, with fixed value for $S_T$, and the evolution of the various CPPIs, full capital guarantee. In (a)-(b) $S_T = 100$, in (c)-(d) $S_T = 200$, in (e) $S_T$=300, in (f) and (g) $S_T = 500$ and, in (h) $S_T = 800$.

Saposnik (1962), and later by Hadar and Russell (1969) and Whitmore (1970) (for higher SD orders), as a more general decision rule than the moment analysis, based on the expected utility maximisation principle. Despite the fact that this framework assumes investors are von Neumann-Morgenstern-rational and maximize expected utility (Linton et al., 2005), SD analysis allows many times for stronger conclusions than the simple and direct analysis of distributions, as it is able to embed in the analysis the typical risk-aversion behaviour of PI investors.

We consider the three orders: first-order SD (FSD) on which is assumed that investors choose only the portfolio with the highest payoff, i.e have utility functions with positive first derivative (Biswas, 2012); second-order SD (SSD) that implies a concave utility function, meaning that risk aversion increases; and the third-order SD (TSD) which requires that investors have convex utility functions, i.e., are risk-seekers when their wealth grows.

In Section 4 we present and discuss our main findings.

# 4 Results

## 4.1 Payoff Distributions

Figures 6 to 13 show the distributions of payoffs associated with the various PI strategies under analysis, for different scenario parameters.

It is worth pointing out once more that these are *conditional* distributions to the extent that we impose a fixed final value to the underlying risky asset at maturity, $S_T$.

In particular for PI strategies are *not* path dependent, i.e. if the payoff does not the depend on the actual evolution, but just on the value of the underlying at maturity, one observes degenerate distributions, with full mass at one point.

As the images below show, the non path-dependent strategies are the OBPI and the CPPI 1. Therefore, CPPI 1 and OBPI distributions are degenerate with a single $100\%$ weighted bar. The OBPI values were computed according to the BS model (recall from Section 2.2 that we use the Synthetic OBPI) while the CPPI 1 payoff at maturity is simply the sum of the insured amount $\eta V_0$ plus the amount invested in the risky asset, i.e. $V_0 - \eta V_0 e^{-rT}$.

The SLPI is path dependent in a very specific way, as its density depends only on whether or not the underlying risk asset touches the floor, so it presents only two possible outcomes. On the other hand, CPPI 3 and CPPI 5 prove to be extremely path dependent in all scenarios.

In the first set of figures (Figures 6–7) we assume full guarantee of capital ($\eta = 100\%$) and an investment period of 5 years ($T = 5$). The different between Figures 6 and 7 is that in the latter we impose a much higher volatility on the underlying risky process, $\sigma = 15\%$ and $\sigma = 40\%$, respectively. One clearly sees that the higher the volatility of the underlying, the higher is the likelihood of CPPI 3 and 5 getting cash-locked.

In Figures 8 and 9 we have increased the investment period to $T = 15$ (as opposed to $T = 5$ in Figures 6 and 7). From the comparison, it is evident that, as the investment period increases the probability of CPPI 3 and CPPI 5 getting cash-locked converges to one and that converge rate is higher the higher the volatility of the underlying asset. Figure 9 shows $100\%$ *probability* of cash-locked for CPPI 3 and 5 for $\sigma = 40\%$, even in the case when the underlying risky asset goes from $S_0 = 100$ to $S_T = 250$.

Tables 1 to 5 summarise the distributions in terms of their first four moments, i.e, the mean, variance, skewness and kurtosis. Actually the second and fourth moments are adjusted to their

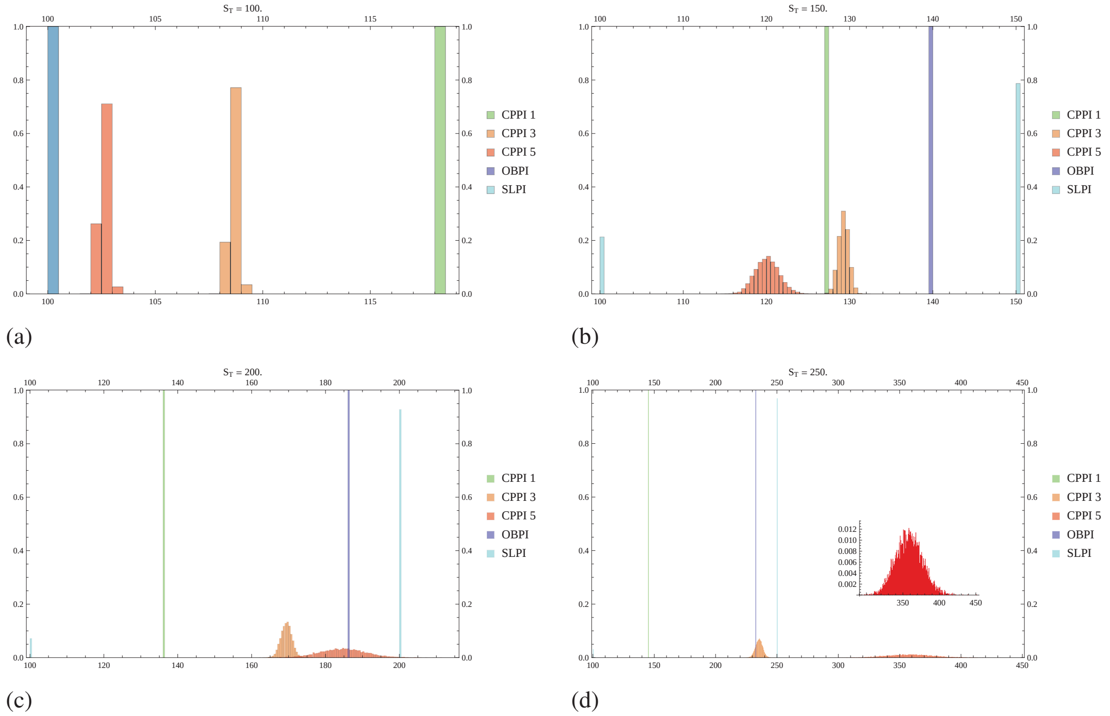



Figure 6: Payoffs at maturity, $\eta = 100\%, T = 5, \sigma = 15\%$

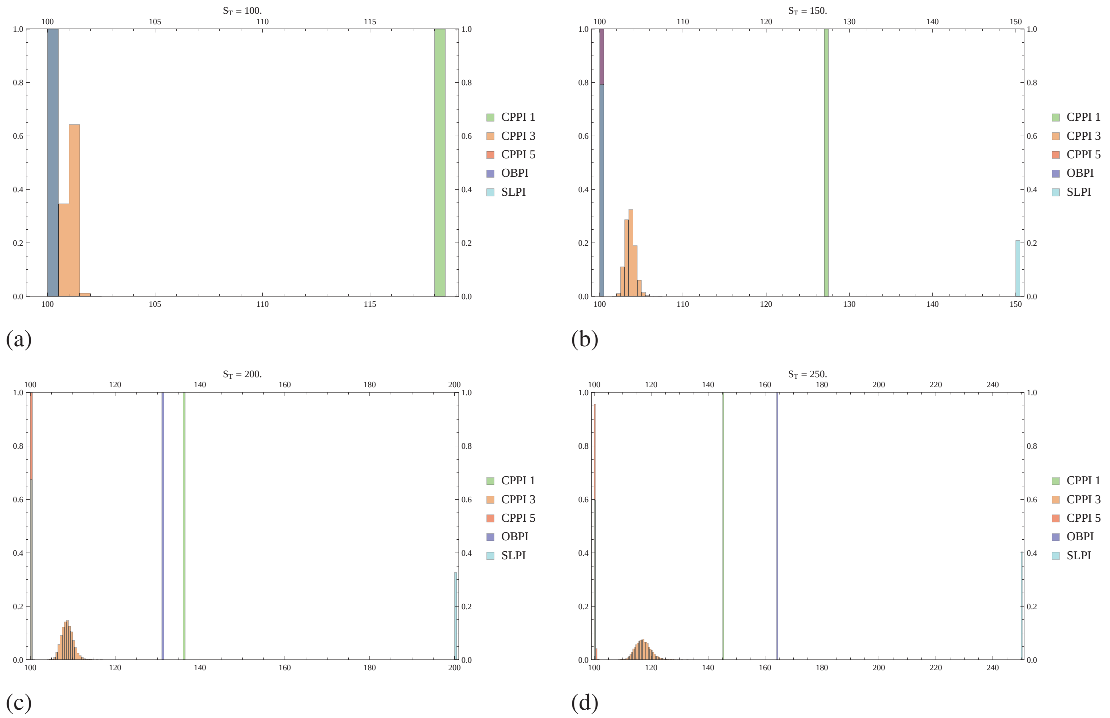
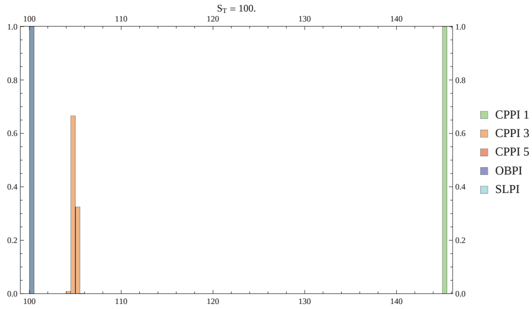
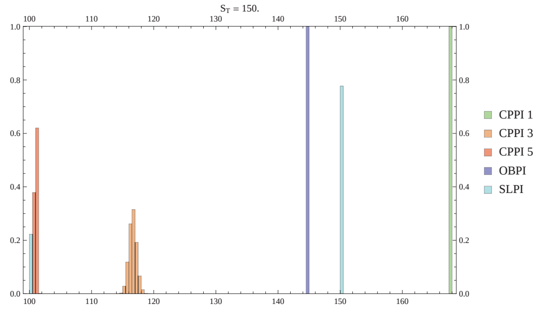


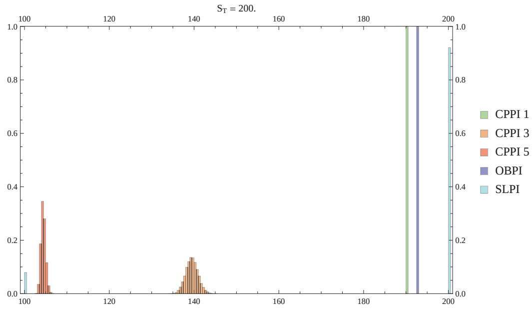

Figure 7: Payoffs at maturity,$\eta = 100\%, T = 5, \sigma = 40\%$



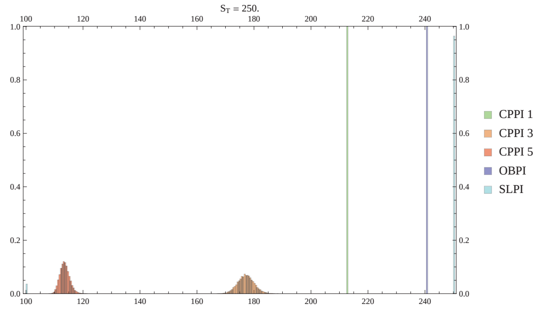

(a)




(b)

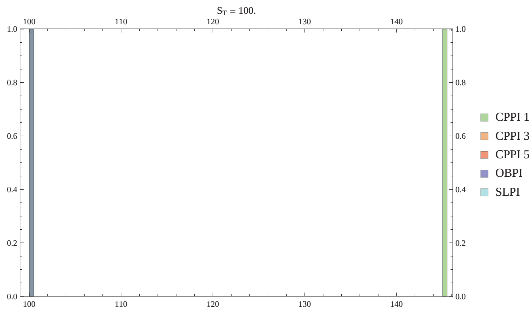



(c)

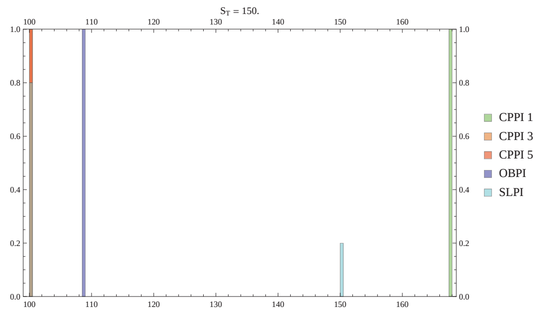



(d)

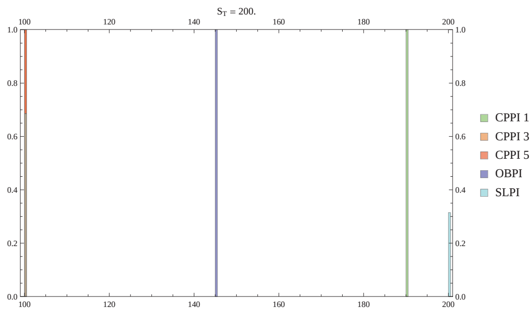
Figure 8: Payoffs at maturity, $\eta = 100\%, T = 15, \sigma = 15\%$



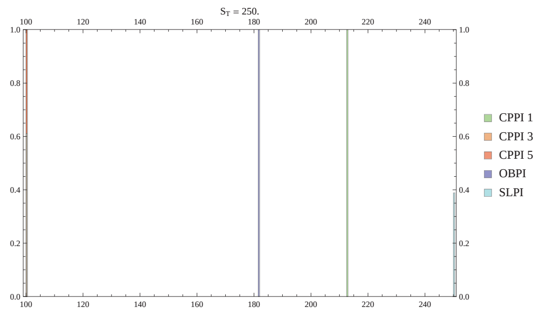

(a)




(b)




(c)




(d)

Figure 9: Payoffs at maturity, $\eta = 100\%, T = 15, \sigma = 40\%$

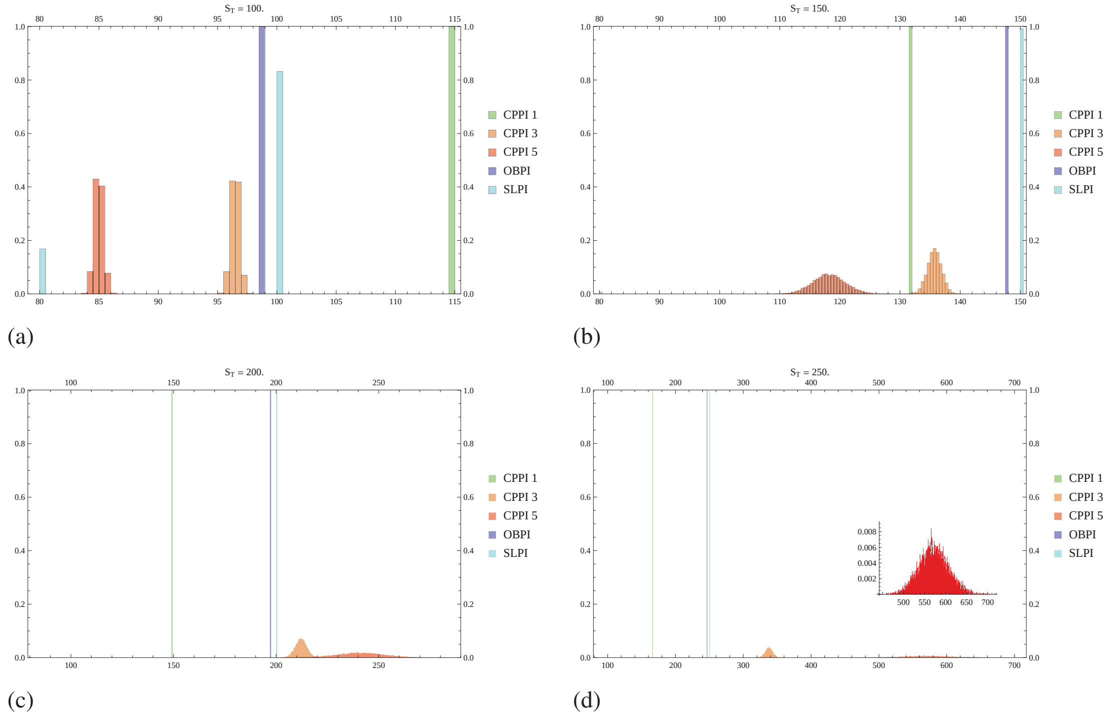



Figure 10: Payoffs at maturity, $\eta = 80\%, T = 5, \sigma = 15\%$

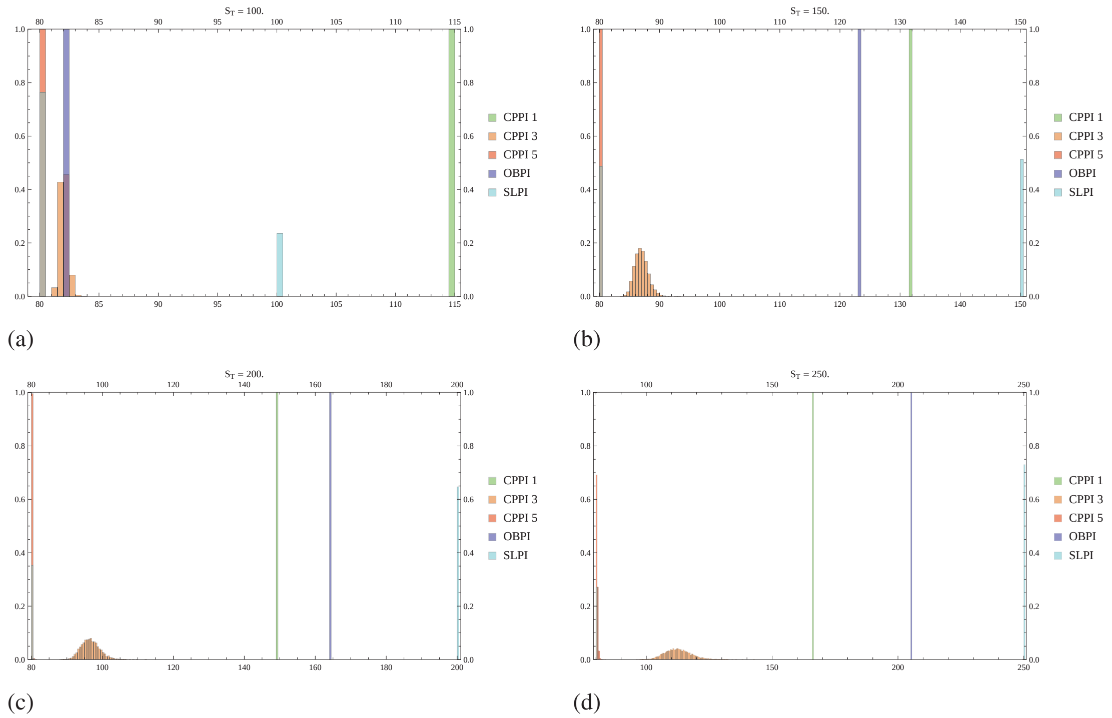



Figure 11: Payoffs at maturity,$\eta = 80\%, T = 5, \sigma = 40\%$

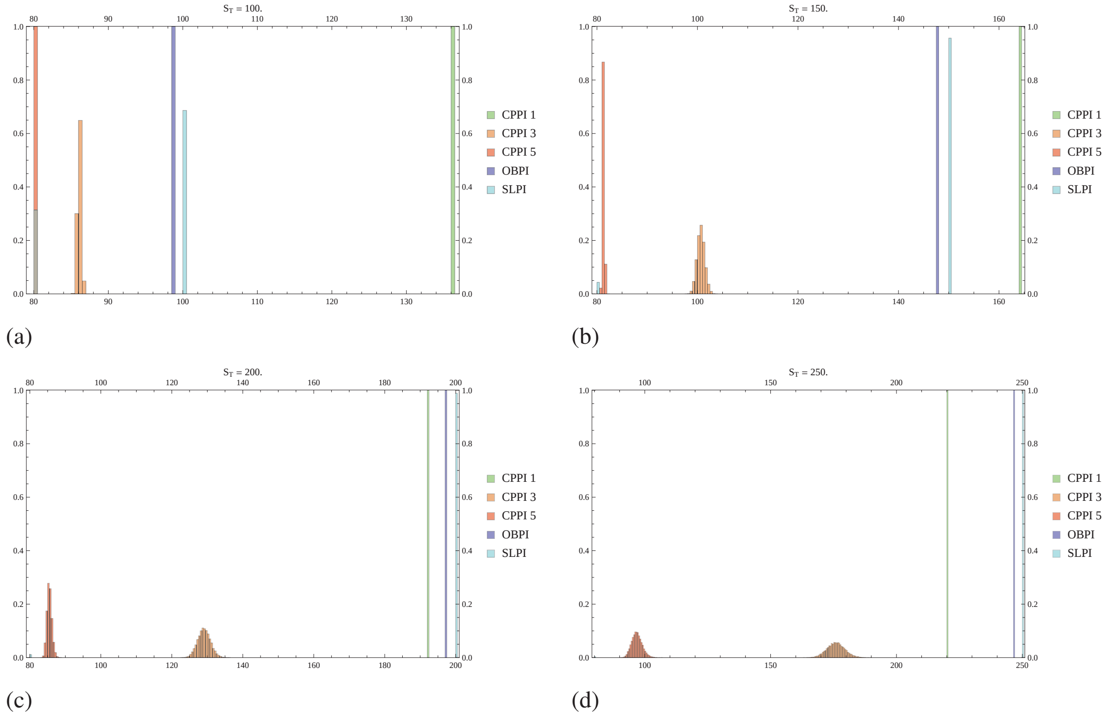



Figure 12: Payoffs at maturity, $\eta = 80\%, T = 15, \sigma = 15\%$

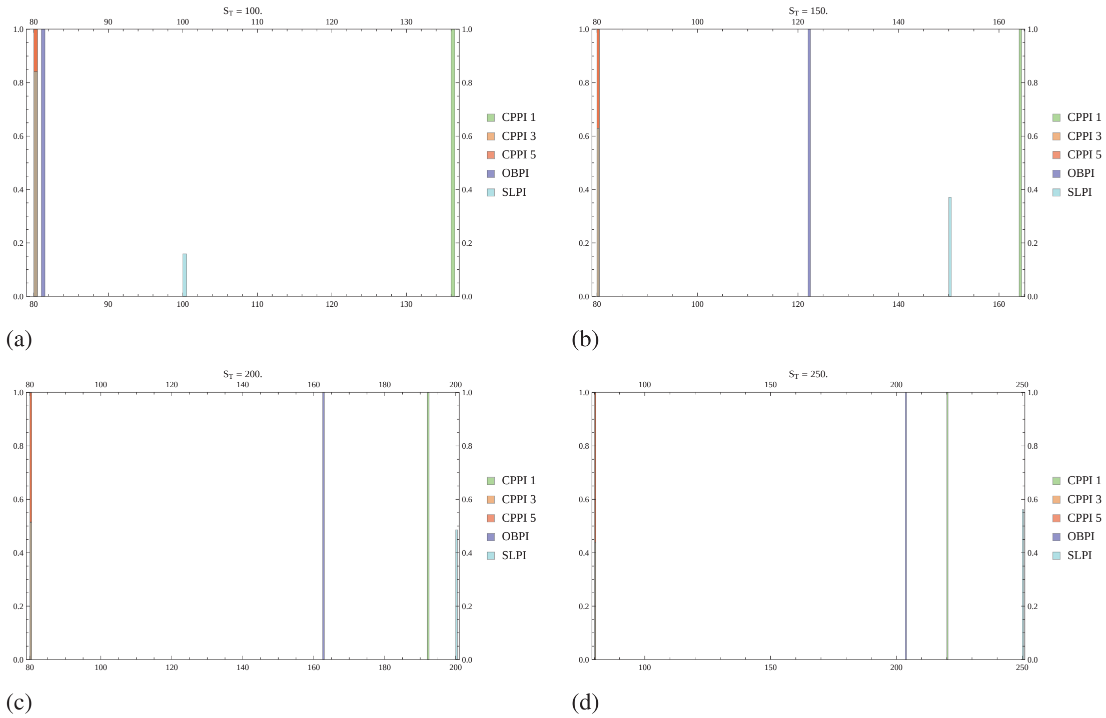



Figure 13: Payoffs at maturity,$\eta = 80\%, T = 15, \sigma = 40\%$

most used and more easily interpretable forms: respectively the standard deviation which is the square root of the variance, and the excess kurtosis which is simply equal to kurtosis - 3. The latter adjustment takes advantage of the fact that normal distribution has 0 kurtosis, hence makes comparison more intuitive. All moments are obtained from the density functions of the portfolio payoffs at maturity, $V_T$, for each PI strategy in all scenarios.

In all tables, the different $S_T$ values displayed on the left column can also be interpreted as the result of a pure Buy and Hold (B&H) strategy with initial investment of $S_0 = 100$. Hence we can compare directly the path-independent strategies with the simplest B&H strategy.

As to the path-dependent strategies – CPPI 3 and CPPI 5 – one must be framed carefully with the other moments.

We begin the moments' analysis with the mean values in Table 1. CPPI 1 strategy mean values do not vary with volatility (path-independence). In general, CPPI 1 exhibits a slight improvement from $\eta = 100\%$ to $\eta = 80\%$ and longer maturity. It outperforms the B&H strategy in the ($S_T = 100$ , $T = 5$) and ($S_T = \{100, 150\}$ , $T = 15$). Moreover, CPPI 1 has a better performance than the OBPI strategy for high volatility and long maturity, but also for $S_T \leq 150$ when ($\sigma = 15\%$, $T = 15$) and ($\sigma = 45\%$, $T = 5$). CPPI 3 is highly dependent on $\sigma$ which is due to its path-dependency. The low mean values for $\sigma = 40\%$ suggest high cash-lock occurrences. While in both volatility cases those occurrences may obviously decrease under higher $S_T$ realisations, only for $\sigma = 15\%$ we can see possible cases of CPPI 3 performing better than B&H in two particular instances ($S_T = 300$ , $\eta = 100\%$) and ($S_T \geq 200$ , $\eta = 80\%$). The CPPI 5 means also show an extreme dependence on the volatility and maturity as cash-lock events may happen for almost every simulation for $\sigma = 40\%$ and $T = 15$. However, for $\sigma = 15\%$ this strategy can outperform B&H not only for ($S_T = 300$ , $\eta = 100\%$) and ($S_T \geq 200$ , $\eta = 80\%$) cases. As for the OBPI we can see that even though it is a path-independent strategy the mean values decrease with volatility because the synthetic OBPI is model dependent, and the European call option prices increase with $\sigma$. Therefore we verify that there no case OBPI is expected to outperforms the B&H, but in low volatile markets it is close to B&H mean values. Also the OBPI average performance if not better than CPPI 1 when $\eta = 15\%$. Finally, the SLPI mean values are very similar to the OBPI, with exception of some cases of $\sigma = 40\%$, but the two strategies are different in respect to path-dependency.

For higher moments, we consider only the path-dependent strategies, CPPI 3 and 5. CPPI 1 and OBPI are obviously left aside because of their degeneracy. SLPI is also path-dependent, but in a different manner, because it has only two possible outcomes: $B_T$ or $S_T$, whichever is the highest at maturity. This means that we do not need the higher order moments to interpret the characteristics of this strategy. All the information is on the probabilities of the two outcomes which are depicted in Table 2. Yet, we still deliver some observations about the skewness and kurtosis of SLPI.

From Table 3 it is very clear that with higher volatility of the risky asset, the standard deviations of the CPPI 3 and CPPI 5 decrease, which may lead to the false interpretation these strategies are "safe". In fact, what these numbers translate is the fact that for high volatility of the risky assets, CPPI 3 and 5 end up very often cash-locked. Similarly, the higher the floor (higher $\eta$) the lower the strategies volatility as the probability of cash-lock events increase. We note that both strategies suffer a decrease in the standard deviation for longer maturities which is enhanced by higher volatilities corroborating the idea that those conditions imply almost sure cash-lock occurrences. We must also emphasise the fact that higher multipliers amplify the negative effect of longer maturities. Finally, for $S_T = 100$ we see that the SLPI distribution is obviously degenerate with

Table 1: Mean of the PI payoff distributions. $V_0 = 100$

| | | $\eta$ = 100% | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\sigma$ = 15% | | | | | $\sigma$ = 40% | | | | |
| | $S_T$ | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| T=5 | 100 | 118.1 | 108.7 | 102.6 | 100.0 | 100.0 | 118.1 | 101.1 | 100.0 | 100.0 | 100.0 |
| | 150 | 127.2 | 129.3 | 120.1 | 139.7 | 139.1 | 127.2 | 103.7 | 100.0 | 100.0 | 110.6 |
| | 200 | 136.3 | 169.4 | 184.8 | 186.3 | 192.4 | 136.3 | 108.7 | 100.1 | 131.5 | 132.8 |
| | 250 | 145.3 | 235.4 | 358.4 | 232.8 | 245.4 | 145.3 | 117.1 | 100.2 | 164.3 | 161.4 |
| | 300 | 154.4 | 333.7 | 741.6 | 279.4 | 296.9 | 154.4 | 129.6 | 100.6 | 197.2 | 193.6 |
| T=15 | 100 | 145.1 | 104.9 | 100.1 | 100.0 | 100.0 | 145.1 | 100.0 | 100.0 | 100.0 | 100.0 |
| | 150 | 167.7 | 116.6 | 101.0 | 144.5 | 138.7 | 167.7 | 100.0 | 100.0 | 108.9 | 109.9 |
| | 200 | 190.2 | 139.5 | 104.4 | 192.7 | 192.1 | 190.2 | 100.1 | 100.0 | 145.2 | 130.6 |
| | 250 | 212.8 | 177.2 | 113.6 | 240.9 | 244.5 | 212.8 | 100.1 | 100.0 | 181.5 | 157.2 |
| | 300 | 235.4 | 233.4 | 133.9 | 289.1 | 296.5 | 235.4 | 100.3 | 100.0 | 217.8 | 187.1 |

| | | $\eta$ = 80% | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\sigma$ = 15% | | | | | $\sigma$ = 40% | | | | |
| | $S_T$ | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| T=5 | 100 | 114.5 | 96.49 | 84.99 | 98.56 | 96.64 | 114.5 | 82.04 | 80.00 | 82.06 | 84.83 |
| | 150 | 131.8 | 135.7 | 118.2 | 147.8 | 149.5 | 131.8 | 86.97 | 80.03 | 123.1 | 116.2 |
| | 200 | 149.0 | 212.0 | 241.3 | 197.1 | 199.9 | 149.0 | 96.61 | 80.14 | 164.1 | 158.2 |
| | 250 | 166.3 | 337.7 | 571.9 | 246.4 | 250.0 | 166.3 | 112.6 | 80.43 | 205.1 | 203.5 |
| | 300 | 183.5 | 524.8 | 1301. | 295.7 | 300.0 | 183.5 | 136.4 | 81.10 | 246.2 | 250.3 |
| T=15 | 100 | 136.1 | 86.12 | 80.17 | 98.62 | 93.58 | 136.1 | 80.01 | 80.00 | 81.41 | 83.11 |
| | 150 | 164.1 | 100.7 | 81.30 | 147.9 | 147.0 | 164.1 | 80.04 | 80.00 | 122.1 | 105.6 |
| | 200 | 192.2 | 129.1 | 85.51 | 197.2 | 198.8 | 192.2 | 80.09 | 80.00 | 162.8 | 137.9 |
| | 250 | 220.2 | 175.9 | 96.88 | 246.5 | 249.6 | 220.2 | 80.19 | 80.00 | 203.5 | 174.6 |
| | 300 | 248.3 | 245.8 | 122.1 | 295.9 | 299.8 | 248.3 | 80.32 | 80.00 | 244.2 | 213.7 |

only one possible outcome 100 because the final floor value coincides with $S_T$.

The *skewness* of a distribution measures its asymmetry with respect to the mean. Specifically, a negative or left-skewed distribution has a longer left tail whereas a distribution with a broader right tail has positive or right skewness. Hence zero-skewed strategies are symmetric. Investors tend to favor positively skewed payoffs, so an analysis merely based on mean and variance measures would overrate the strategies which reduce skewness. In Table 4 we can see that for CPPI 3 and 5, $\eta$ does not influence skewness (not even kurtosis as can see ahead) but the increasing volatility makes distributions more positive-skewed. In addition higher $S_T$ values give place to very small decreases in skewness while longer $T$ gives more positive skewness. For the SLPI skewness along with the mean values show the bimodal aspect of the distribution. For $\sigma = 15\%$ it is always

Table 2: SLPI probabilities.

| | | $\eta$=100% | | | | $\eta$=80% | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $\sigma$=15% | | $\sigma$=40% | | $\sigma$=15% | | $\sigma$=40% | |
| | ST | $S_T$ | $F_T$ | $S_T$ | $F_T$ | $S_T$ | $F_T$ | $S_T$ | $F_T$ |
| T=5 | 100 | 1. | 1. | 1. | 1. | 0.8319 | 0.1681 | 0.2414 | 0.7586 |
| | 150 | 0.7819 | 0.2181 | 0.211 | 0.789 | 0.9924 | 0.0076 | 0.5178 | 0.4822 |
| | 200 | 0.9243 | 0.0757 | 0.3277 | 0.6723 | 0.9993 | 0.0007 | 0.6518 | 0.3482 |
| | 250 | 0.969 | 0.031 | 0.4096 | 0.5904 | 0.9998 | 0.0002 | 0.7267 | 0.2733 |
| | 300 | 0.9843 | 0.0157 | 0.468 | 0.532 | 0.9999 | 0.0001 | 0.7739 | 0.2261 |
| T=15 | 100 | 1. | 1. | 1. | 1. | 0.679 | 0.321 | 0.1553 | 0.8447 |
| | 150 | 0.7744 | 0.2256 | 0.1971 | 0.8029 | 0.9566 | 0.0434 | 0.3653 | 0.6347 |
| | 200 | 0.9214 | 0.0786 | 0.3063 | 0.6937 | 0.9903 | 0.0097 | 0.4823 | 0.5177 |
| | 250 | 0.9633 | 0.0367 | 0.3815 | 0.6185 | 0.9977 | 0.0023 | 0.5565 | 0.4435 |
| | 300 | 0.9823 | 0.0177 | 0.4354 | 0.5646 | 0.9989 | 0.0011 | 0.6077 | 0.3923 |

Table 3: Standard Deviation of the PI payoff distributions. $V_0 = 100$

| | $\eta$ = 100% | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\sigma$ = 15% | | | | | $\sigma$ = 40% | | | | |
| | $S_T$ | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| T=5 | 100 | 0 | 0.1854 | 0.1892 | 0 | 0 | 0 | 0.1684 | 0.001230 | 0 | 0 |
| | 150 | 0 | 0.6242 | 1.438 | 0 | 20.65 | 0 | 0.5715 | 0.009846 | 0 | 20.40 |
| | 200 | 0 | 1.475 | 6.039 | 0 | 26.45 | 0 | 1.358 | 0.04289 | 0 | 46.94 |
| | 250 | 0 | 2.872 | 18.34 | 0 | 26.00 | 0 | 2.657 | 0.1340 | 0 | 73.77 |
| | 300 | 0 | 4.948 | 45.38 | 0 | 24.86 | 0 | 4.595 | 0.3393 | 0 | 99.80 |
| T=15 | 100 | 0 | 0.1816 | 0.01701 | 0 | 0 | 0 | 0.002558 | $1.388\times10^{-6}$ | 0 | 0 |
| | 150 | 0 | 0.6135 | 0.1302 | 0 | 20.90 | 0 | 0.008712 | $6.225\times10^{-10}$ | 0 | 19.89 |
| | 200 | 0 | 1.454 | 0.5510 | 0 | 26.91 | 0 | 0.02078 | $2.735\times10^{-9}$ | 0 | 46.10 |
| | 250 | 0 | 2.840 | 1.686 | 0 | 28.21 | 0 | 0.04076 | $8.613\times10^{-9}$ | 0 | 72.87 |
| | 300 | 0 | 4.906 | 4.201 | 0 | 26.37 | 0 | 0.07068 | $2.198\times10^{-8}$ | 0 | 99.17 |

| | $\eta$ = 80% | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\sigma$ = 15% | | | | | $\sigma$ = 40% | | | | |
| | $S_T$ | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| T=5 | 100 | 0 | 0.3530 | 0.3601 | 0 | 7.479 | 0 | 0.3205 | 0.002341 | 0 | 8.559 |
| | 150 | 0 | 1.188 | 2.737 | 0 | 6.080 | 0 | 1.088 | 0.01874 | 0 | 34.98 |
| | 200 | 0 | 2.807 | 11.49 | 0 | 3.174 | 0 | 2.586 | 0.08164 | 0 | 57.17 |
| | 250 | 0 | 5.466 | 34.91 | 0 | 2.404 | 0 | 5.058 | 0.2550 | 0 | 75.76 |
| | 300 | 0 | 9.417 | 86.38 | 0 | 2.200 | 0 | 8.746 | 0.6457 | 0 | 92.03 |
| T=15 | 100 | 0 | 0.2258 | 0.02114 | 0 | 9.338 | 0 | 0.003180 | $1.725\times10^{-6}$ | 0 | 7.244 |
| | 150 | 0 | 0.7627 | 0.1619 | 0 | 14.26 | 0 | 0.01083 | $7.739\times10^{-10}$ | 0 | 33.71 |
| | 200 | 0 | 1.808 | 0.6850 | 0 | 11.76 | 0 | 0.02583 | $3.400\times10^{-9}$ | 0 | 59.97 |
| | 250 | 0 | 3.531 | 2.096 | 0 | 8.144 | 0 | 0.05068 | $1.071\times10^{-8}$ | 0 | 84.46 |
| | 300 | 0 | 6.099 | 5.223 | 0 | 7.293 | 0 | 0.08788 | $2.732\times10^{-8}$ | 0 | 107.4 |

left skewed (with exception of $S_T = 100$) because there were more $V_T = S_T$ realisations than $V_T = B_T$ conferring an effective left tail to the distribution. For higher $S_T$ values the left-skewness intensifies because there are less chances of triggering the stop-loss rule and therefore more weight is given on the right bar.

The exact interpretation of *tailedness* and *peakedness* of a distribution function provided by the kurtosis has been subject to wide discussion (and often confusion) over the past century (DeCarlo, 1997). Yet presently there is still room for presumptions that can give alternative measurements of a distributions peak sharpness and tail fatness, because different shaped distributions with equal kurtosis have been already found. However it is consensual that shape has to incorporate those two aspects (peak and tails). Therefore the kurtosis measurement basically assumes that the shoulders of a distribution are located at the mean plus (and minus) a standard deviation and scales the fourth moment to its variance. Another common meaning used for kurtosis is the 'departure from normality". Hence, normal/mesokurtic distributions have excess kurtosis $\gamma_2 = 0$ (or 3 for kurtosis), $\gamma_2 > 0$ correspond to leptokurtic curves, i.e., with sharp peak and fat tails, while platykurtic shapes measure $\gamma_2 < 0$, are flat at the peak and have short tails. This being said it can be observed in Table 5 the same independence on $\eta$ as in the skewness values. CPPI 3 and 5 are always leptokurtic but almost normal for $\sigma = 15\%$ and CPPI 3 has still low positive kurtosis for $\sigma = 45\%$. However CPPI 5 bypasses positively the normal range for high $\sigma = 45\%$, but even more heavily when adding long maturities large $\gamma_2$. For the SLPI strategy again kurtosis shows a different behaviour. In general, for low $S_T$ the two possible outcome bars are more close and equitably distributed hence decreasing the absolute value of skewness and kurtosis. Has $S_T$ rises, the left bar stays fixed and the right bard increasingly detaches from the other as it gains more weight simultaneously.

Table 4: Skewness (third moment).

| | $\eta$ = 100% | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\sigma$ = 15% | | | | | $\sigma$ = 40% | | | | |
| | $S_T$ | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| T=5 | 100 | Ind | −0.06993 | 0.08492 | Ind | Ind | Ind | 0.3413 | 1.797 | Ind | Ind |
| | 150 | Ind | −0.07020 | 0.08331 | Ind | −1.365 | Ind | 0.3393 | 1.777 | Ind | 1.417 |
| | 200 | Ind | −0.07039 | 0.08217 | Ind | −3.208 | Ind | 0.3379 | 1.763 | Ind | 0.7342 |
| | 250 | Ind | −0.07053 | 0.08129 | Ind | −5.412 | Ind | 0.3368 | 1.752 | Ind | 0.3677 |
| | 300 | Ind | −0.07065 | 0.08058 | Ind | −7.792 | Ind | 0.3359 | 1.743 | Ind | 0.1283 |
| T=15 | 100 | Ind | 0.04592 | 0.3100 | Ind | Ind | Ind | 0.7670 | −99.98 | Ind | Ind |
| | 150 | Ind | 0.04577 | 0.3091 | Ind | −1.313 | Ind | 0.7657 | 4.696 | Ind | 1.523 |
| | 200 | Ind | 0.04567 | 0.3084 | Ind | −3.132 | Ind | 0.7648 | 4.679 | Ind | 0.8404 |
| | 250 | Ind | 0.04559 | 0.3079 | Ind | −4.928 | Ind | 0.7642 | 4.666 | Ind | 0.4879 |
| | 300 | Ind | 0.04552 | 0.3075 | Ind | −7.315 | Ind | 0.7636 | 4.654 | Ind | 0.2606 |

| | $\eta$ = 80% | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\sigma$ = 15% | | | | | $\sigma$ = 40% | | | | |
| | $S_T$ | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| T=5 | 100 | Ind | −0.06993 | 0.08492 | Ind | −1.775 | Ind | 0.3413 | 1.797 | Ind | 1.209 |
| | 150 | Ind | −0.07020 | 0.08331 | Ind | −11.34 | Ind | 0.3393 | 1.777 | Ind | −0.07125 |
| | 200 | Ind | −0.07039 | 0.08217 | Ind | −37.76 | Ind | 0.3379 | 1.763 | Ind | −0.6373 |
| | 250 | Ind | −0.07053 | 0.08129 | Ind | −70.69 | Ind | 0.3368 | 1.752 | Ind | −1.017 |
| | 300 | Ind | −0.07065 | 0.08058 | Ind | −99.98 | Ind | 0.3359 | 1.743 | Ind | −1.310 |
| T=15 | 100 | Ind | 0.04592 | 0.3100 | Ind | −0.7668 | Ind | 0.7670 | −99.98 | Ind | 1.903 |
| | 150 | Ind | 0.04577 | 0.3091 | Ind | −4.482 | Ind | 0.7657 | 4.696 | Ind | 0.5595 |
| | 200 | Ind | 0.04567 | 0.3084 | Ind | −10.01 | Ind | 0.7648 | 4.679 | Ind | 0.07084 |
| | 250 | Ind | 0.04559 | 0.3079 | Ind | −20.78 | Ind | 0.7642 | 4.665 | Ind | −0.2275 |
| | 300 | Ind | 0.04552 | 0.3075 | Ind | −30.10 | Ind | 0.7636 | 4.654 | Ind | −0.4412 |

Table 5: Excess Kurtosis (fourth moment −3).

| | $\eta$ = 100% | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\sigma$ = 15% | | | | | $\sigma$ = 40% | | | | |
| | $S_T$ | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| T=5 | 100 | Ind | 0.06056 | 0.06018 | Ind | Ind | Ind | 0.2492 | 6.003 | Ind | Ind |
| | 150 | Ind | 0.06063 | 0.05974 | Ind | −0.1360 | Ind | 0.2468 | 5.866 | Ind | 0.006764 |
| | 200 | Ind | 0.06069 | 0.05943 | Ind | 8.292 | Ind | 0.2451 | 5.771 | Ind | −1.461 |
| | 250 | Ind | 0.06073 | 0.05920 | Ind | 27.29 | Ind | 0.2438 | 5.699 | Ind | −1.865 |
| | 300 | Ind | 0.06076 | 0.05902 | Ind | 58.71 | Ind | 0.2428 | 5.641 | Ind | −1.984 |
| T=15 | 100 | Ind | 0.02965 | 0.1870 | Ind | Ind | Ind | 1.073 | 9995. | Ind | Ind |
| | 150 | Ind | 0.02963 | 0.1859 | Ind | −0.2761 | Ind | 1.069 | 42.37 | Ind | 0.3191 |
| | 200 | Ind | 0.02962 | 0.1852 | Ind | 7.808 | Ind | 1.067 | 42.06 | Ind | −1.294 |
| | 250 | Ind | 0.02961 | 0.1847 | Ind | 22.29 | Ind | 1.065 | 41.82 | Ind | −1.762 |
| | 300 | Ind | 0.02960 | 0.1842 | Ind | 51.52 | Ind | 1.063 | 41.63 | Ind | −1.932 |

| | $\eta$ = 80% | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\sigma$ = 15% | | | | | $\sigma$ = 40% | | | | |
| | $S_T$ | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| T=5 | 100 | Ind | 0.06056 | 0.06018 | Ind | 1.151 | Ind | 0.2492 | 6.003 | Ind | −0.5393 |
| | 150 | Ind | 0.06063 | 0.05974 | Ind | 126.6 | Ind | 0.2468 | 5.866 | Ind | −1.995 |
| | 200 | Ind | 0.06069 | 0.05943 | Ind | 1424. | Ind | 0.2451 | 5.771 | Ind | −1.594 |
| | 250 | Ind | 0.06073 | 0.05920 | Ind | 4995. | Ind | 0.2438 | 5.699 | Ind | −0.9649 |
| | 300 | Ind | 0.06076 | 0.05902 | Ind | 9995. | Ind | 0.2428 | 5.641 | Ind | −0.2850 |
| T=15 | 100 | Ind | 0.02965 | 0.1870 | Ind | −1.412 | Ind | 1.073 | 9995. | Ind | 1.623 |
| | 150 | Ind | 0.02963 | 0.1859 | Ind | 18.09 | Ind | 1.069 | 42.37 | Ind | −1.687 |
| | 200 | Ind | 0.02962 | 0.1852 | Ind | 98.10 | Ind | 1.067 | 42.06 | Ind | −1.995 |
| | 250 | Ind | 0.02961 | 0.1847 | Ind | 429.8 | Ind | 1.065 | 41.82 | Ind | −1.948 |
| | 300 | Ind | 0.02960 | 0.1842 | Ind | 904.1 | Ind | 1.063 | 41.63 | Ind | −1.805 |

## 4.2 Stochastic Dominance

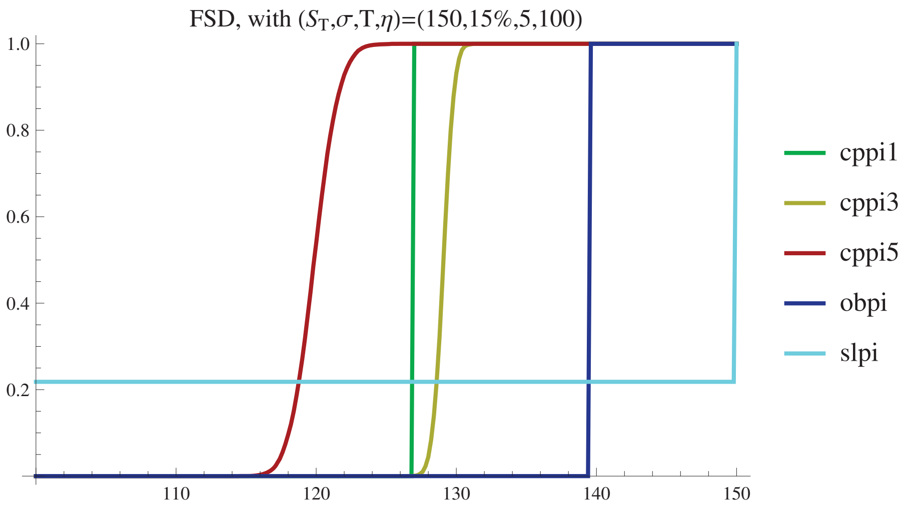
Consider two random variables $V_1$ and $V_2$, and their respective cumulative distribution functions (CDF), $F_1(x)$ and $F_2(x)$. Then, we say that $V_1$ $i$th order stochastically dominates $V_2$ if and only if $D_1^{(i)}(x) \leq D_2^{(i)}(x)\ \forall x$ (with strict inequality for at least one $x$), where $D_k^{(i)} = \int_{-\infty}^{x} D_k^{(i-1)} dx$ and $D_k^{(1)} = F_k(x)$ (Davidson and Duclos, 2000; Annaert et al., 2009). We denote $V_1$ stochastically dominates $V_2$ on first (second and third) order by $V_1$ FSD (respectively SSD, TSD) $V_2$ (as in e.g. Levy and Wiener (1998)). Therefore, if the CDF of the two strategies intersect or are equal, there is no SD between them. The test is made in both directions because if $V_1$ does not SD $V_2$, it does not mean that $V_2$ SD $V_1$. Contrarily, it is obvious that if $V_1$ SD $V_2$ we know the reverse does not. Therefore this study organizes the stochastic dominance results so that no duplications arise. In addition, successive narrowing of the class of utility functions contemplated on higher order SD suggests that lower degree SD imply necessarily the SD on the subsequent orders, i.e., FSD⇒SSD⇒TSD.

Figure 14 illustrates an example of three orders of stochastic dominance in a scenario described in the caption bellow. Tables 6 – 6 summarise the identified dominances. The observations on (F, S and T)SD (see Tables (6, 7 and 8) correspondingly) are made separately but the higher the order, the less observations since the rest are resumed in the lower order SD.

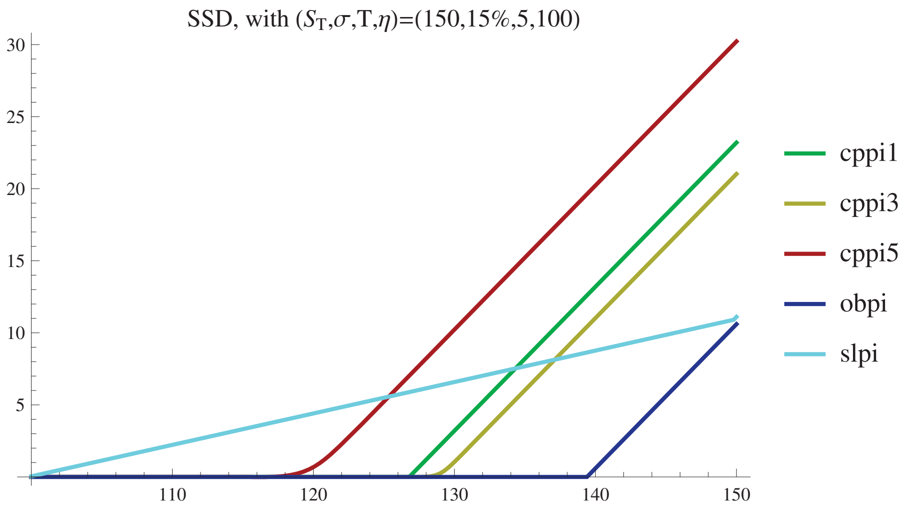
For first order SD, investors who are concerned simply with higher payoff, prefer always CPPI 1 to all other strategies for $S_T = 100$ in every scenario and for $(ST = 150, T = 15)$, confirming the mean analysis. It also FSD CPPI 3 and 5 in all scenarios except for $\sigma = 15\%$, $(S_T \geq 150, T = 5)$ and $(S_T = 300, T = 15)$. The choices of insurance percentage generally do not influence CPPI 1's dominance, existing only one exception, where the dominance over OBPI in $(S_T = 200, \sigma = 40\%, T = 5)$ and $\eta = 100\%$ is lost for $\eta = 80\%$. CPPI 3 FSD all strategies except CPPI 1 for the lowest $S_T$ and $\eta = 100\%$ in all volatility and maturity cases. It also FSD CPPI 5 in every scenario except for $(S_T \geq 250, \sigma = 15\%, T = 5)$ which are the only cases it dominates CPPI 1. CPPI 5 FSD all strategies for $(S_T \geq 250, \sigma = 15\%, T = 5)$ for both floor choices. Also dominates on first order OBPI and SLPI for $S_T = 100$. OBPI dominates all strategies except SLPI for most cases where $S_T \geq 200$ except when CPPI 3 and 5 dominate. For $S_T \geq 150$ it also presents some dominance on low volatile markets. SLPI first order SD CPPI 3 and 5 for high volatility markets and long maturity.

In respect to second order stochastic dominance, the investors who are risk averse would choose CPPI 1 over SLPI in some cases of high volatile markets, such as for $(S_T = 150, T = 5)$ and for $(S_T \geq 200, T = 15)$ for both $\eta$. CPPI 1 also dominates CPPI 3 on second order for $(S_T = 300, \sigma = 15\%, T = 15)$ for both insurance percentages as well. OBPI dominates CPPI 3 only in two very different cases: $(\sigma = 40\%, T = 5)$ and $(\sigma = 15\%, T = 15)$ in both cases for $S_T = 100$ and $\eta = 80\%$.

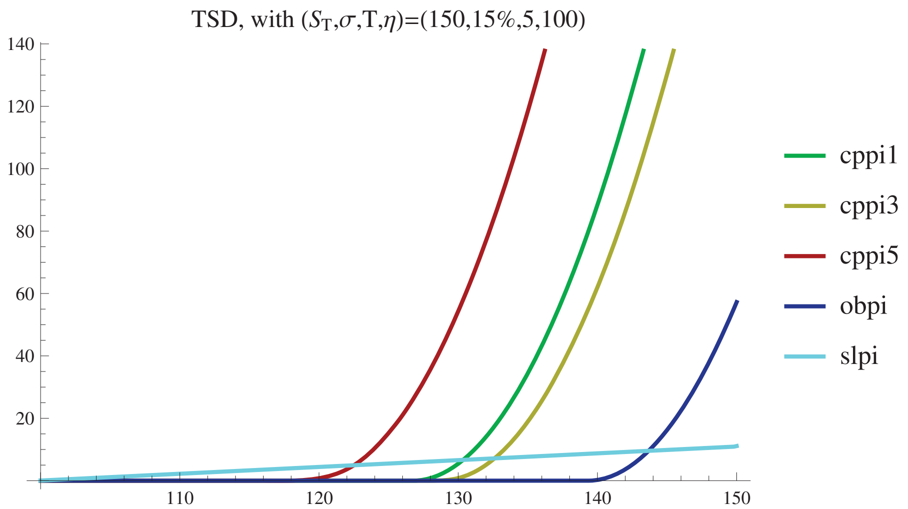
Concerning third order stochastic dominance, the investor whose risk aversion decreases with growing wealth, chooses CPPI 1 over SLPI and OBPI in few cases of high volatility with long maturity, or low volatility with short maturity, but both cases for $S_T = \{200, 250\}$. CPPI 3 and 5 are also preferable to this investor than SLPI for some cases of low volatility and $T = 5$: the first strategy for $(S_T = 150, \eta = 80\%)$ and $(S_T = 100, \eta = 80\%)$, and the second for $(S_T = 200, \eta = 100\%)$. Finally, OBPI also stochastically dominates on third order the SLPI strategy for $S_T \geq 150$ and $(\sigma = 15\%, T = 15)$.




(a)




(b)




(c)

Figure 14: First, second and third orders of stochastic dominances: (a) CDF, (b)$\int CDF$ (c) $\int\int CDF$. Scenario: $\{S_T, \sigma, T, \eta\} = \{150, 15\%, 5, 100\%\}$

Table 6: First order stochastic dominance.

| ST | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
|---|---|---|---|---|---|
| | | | $\eta$ = 100% , $\sigma$ = 15% , T = 5 | | |
| 100 | All | cppi5,obpi,slpi | obpi,slpi | slpi | None |
| 150 | cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| 200 | None | cppi1 | cppi1 | cppi1,cppi3 | None |
| 250 | None | cppi1 | All | cppi1 | None |
| 300 | None | cppi1,obpi,slpi | All | cppi1 | None |
| | | | $\eta$ = 100% , $\sigma$ = 40% , T = 5 | | |
| ST | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| 100 | All | cppi5,obpi,slpi | obpi,slpi | slpi | None |
| 150 | cppi3,cppi5,obpi | cppi5,obpi | obpi | None | cppi5,obpi |
| 200 | cppi3,cppi5,obpi | cppi5 | None | cppi3,cppi5 | cppi5 |
| 250 | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | cppi5 |
| 300 | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| | | | $\eta$ = 100% , $\sigma$ = 15% , T = 15 | | |
| ST | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| 100 | All | cppi5,obpi,slpi | obpi,slpi | slpi | None |
| 150 | All | cppi5 | None | cppi3,cppi5 | None |
| 200 | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| 250 | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| 300 | cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| | | | $\eta$ = 100% , $\sigma$ = 40% , T = 15 | | |
| ST | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| 100 | All | cppi5,obpi,slpi | obpi,slpi | slpi | None |
| 150 | All | cppi5 | None | cppi3,cppi5 | cppi3,cppi5 |
| 200 | cppi3,cppi5,obpi | cppi5 | None | cppi3,cppi5 | cppi3,cppi5 |
| 250 | cppi3,cppi5,obpi | cppi5 | None | cppi3,cppi5 | cppi3,cppi5 |
| 300 | cppi3,cppi5,obpi | cppi5 | None | cppi3,cppi5 | cppi3,cppi5 |
| | | | $\eta$ = 80% , $\sigma$ = 15% , T = 5 | | |
| ST | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| 100 | All | cppi5 | None | cppi3,cppi5 | None |
| 150 | cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| 200 | None | cppi1,obpi,slpi | cppi1,obpi,slpi | cppi1 | None |
| 250 | None | cppi1,obpi,slpi | All | cppi1 | None |
| 300 | None | cppi1,obpi,slpi | All | cppi1 | None |
| | | | $\eta$ = 80% , $\sigma$ = 40% , T = 5 | | |
| ST | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| 100 | All | cppi5 | None | cppi5 | cppi5 |
| 150 | cppi3,cppi5,obpi | cppi5 | None | cppi3,cppi5 | cppi5 |
| 200 | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | cppi5 |
| 250 | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | cppi5 |
| 300 | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| | | | $\eta$ = 80% , $\sigma$ = 15% , T = 15 | | |
| ST | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| 100 | All | cppi5 | None | cppi3,cppi5 | cppi5 |
| 150 | All | cppi5 | None | cppi3,cppi5 | None |
| 200 | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| 250 | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| 300 | cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| | | | $\eta$ = 80% , $\sigma$ = 40% , T = 15 | | |
| ST | CPPI1 | CPPI3 | CPPI5 | OBPI | SLPI |
| 100 | All | cppi5 | None | cppi3,cppi5 | cppi3,cppi5 |
| 150 | All | cppi5 | None | cppi3,cppi5 | cppi3,cppi5 |
| 200 | cppi3,cppi5,obpi | cppi5 | None | cppi3,cppi5 | cppi3,cppi5 |
| 250 | cppi3,cppi5,obpi | cppi5 | None | cppi3,cppi5 | cppi3,cppi5 |
| 300 | cppi3,cppi5,obpi | cppi5 | None | cppi3,cppi5 | cppi3,cppi5 |

Table 7: Second order stochastic dominance.

| | | $\eta$ = 100% , $\sigma$ = 15% , T = 5 | | | |
|---|---|---|---|---|---|
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5,obpi,slpi | obpi,slpi | slpi | None |
| **150** | cppi5 | cppi5 | None | All | None |
| **200** | None | cppi1 | cppi1 | cppi1,cppi3,cppi5 | None |
| **250** | None | cppi1 | All | cppi1 | None |
| **300** | None | cppi1,obpi,slpi | All | cppi1 | None |
| | | $\eta$ = 100% , $\sigma$ = 40% , T = 5 | | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5,obpi,slpi | obpi,slpi | slpi | None |
| **150** | All | cppi5,obpi | obpi | None | cppi5,obpi |
| **200** | All | cppi5 | None | cppi3,cppi5 | cppi5 |
| **250** | cppi3,cppi5 | cppi5 | None | All | cppi5 |
| **300** | cppi3,cppi5 | cppi5 | None | All | None |
| | | $\eta$ = 100% , $\sigma$ = 15% , T = 15 | | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5,obpi,slpi | obpi,slpi | slpi | None |
| **150** | All | cppi5 | None | cppi3,cppi5,slpi | None |
| **200** | cppi3,cppi5 | cppi5 | None | All | None |
| **250** | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| **300** | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| | | $\eta$ = 100% , $\sigma$ = 40% , T = 15 | | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5,obpi,slpi | obpi,slpi | slpi | None |
| **150** | All | cppi5 | None | cppi3,cppi5 | cppi3,cppi5 |
| **200** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| **250** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| **300** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| | | $\eta$ = 80% , $\sigma$ = 15% , T = 5 | | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5 | None | cppi3,cppi5,slpi | None |
| **150** | cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| **200** | None | cppi1,obpi,slpi | cppi1,obpi,slpi | cppi1 | None |
| **250** | None | cppi1,obpi,slpi | All | cppi1 | None |
| **300** | None | cppi1,obpi,slpi | All | cppi1 | None |
| | | $\eta$ = 80% , $\sigma$ = 40% , T = 5 | | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5 | None | cppi3,cppi5 | cppi5 |
| **150** | All | cppi5 | None | cppi3,cppi5,slpi | cppi5 |
| **200** | cppi3,cppi5 | cppi5 | None | All | cppi5 |
| **250** | cppi3,cppi5 | cppi5 | None | All | cppi5 |
| **300** | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| | | $\eta$ = 80% , $\sigma$ = 15% , T = 15 | | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5 | None | cppi3,cppi5,slpi | cppi5 |
| **150** | All | cppi5 | None | cppi3,cppi5,slpi | None |
| **200** | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| **250** | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| **300** | cppi3,cppi5 | cppi5 | None | cppi1,cppi3,cppi5 | None |
| | | $\eta$ = 80% , $\sigma$ = 40% , T = 15 | | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5 | None | cppi3,cppi5 | cppi3,cppi5 |
| **150** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| **200** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| **250** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| **300** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |

Table 8: Third order stochastic dominance.

| | | | | | |
|---|---|---|---|---|---|
| | | | **$\eta$ = 100% , $\sigma$ = 15% , T = 5** | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5,obpi,slpi | obpi,slpi | slpi | None |
| **150** | cppi5 | cppi5,slpi | None | All | None |
| **200** | None | cppi1 | cppi1,slpi | All | None |
| **250** | None | cppi1 | All | cppi1 | None |
| **300** | None | cppi1,obpi,slpi | All | cppi1 | None |
| | | | **$\eta$ = 100% , $\sigma$ = 40% , T = 5** | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5,obpi,slpi | obpi,slpi | slpi | None |
| **150** | All | cppi5,obpi | obpi | None | cppi5,obpi |
| **200** | All | cppi5 | None | cppi3,cppi5,slpi | cppi5 |
| **250** | cppi3,cppi5,slpi | cppi5 | None | All | cppi5 |
| **300** | cppi3,cppi5 | cppi5 | None | All | None |
| | | | **$\eta$ = 100% , $\sigma$ = 15% , T = 15** | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5,obpi,slpi | obpi,slpi | slpi | None |
| **150** | All | cppi5 | None | cppi3,cppi5,slpi | None |
| **200** | cppi3,cppi5,slpi | cppi5 | None | All | None |
| **250** | cppi3,cppi5 | cppi5 | None | All | None |
| **300** | cppi3,cppi5 | cppi5 | None | All | None |
| | | | **$\eta$ = 100% , $\sigma$ = 40% , T = 15** | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5,obpi,slpi | obpi,slpi | slpi | None |
| **150** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| **200** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| **250** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| **300** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| | | | **$\eta$ = 80% , $\sigma$ = 15% , T = 5** | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5,slpi | None | cppi3,cppi5,slpi | None |
| **150** | cppi5 | cppi5 | None | All | None |
| **200** | None | cppi1,obpi,slpi | cppi1,obpi,slpi | cppi1 | None |
| **250** | None | cppi1,obpi,slpi | All | cppi1 | None |
| **300** | None | cppi1,obpi,slpi | All | cppi1 | None |
| | | | **$\eta$ = 80% , $\sigma$ = 40% , T = 5** | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5 | None | cppi3,cppi5 | cppi5 |
| **150** | All | cppi5 | None | cppi3,cppi5,slpi | cppi5 |
| **200** | cppi3,cppi5,slpi | cppi5 | None | All | cppi5 |
| **250** | cppi3,cppi5,slpi | cppi5 | None | All | cppi5 |
| **300** | cppi3,cppi5 | cppi5 | None | All | None |
| | | | **$\eta$ = 80% , $\sigma$ = 15% , T = 15** | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5 | None | cppi3,cppi5,slpi | cppi5 |
| **150** | All | cppi5 | None | cppi3,cppi5,slpi | None |
| **200** | cppi3,cppi5,slpi | cppi5 | None | All | None |
| **250** | cppi3,cppi5 | cppi5 | None | All | None |
| **300** | cppi3,cppi5 | cppi5 | None | All | None |
| | | | **$\eta$ = 80% , $\sigma$ = 40% , T = 15** | | |
| **ST** | **CPPI1** | **CPPI3** | **CPPI5** | **OBPI** | **SLPI** |
| **100** | All | cppi5 | None | cppi3,cppi5 | cppi3,cppi5 |
| **150** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| **200** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| **250** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |
| **300** | All | cppi5 | None | cppi3,cppi5,slpi | cppi3,cppi5 |

## 4.3 Discussion of Main Results

We now come to the selection and discussion of the most important results presented above. Our results allow us to make some important conclusions about the path-(in)dependent behaviour of each studied PI. Taking into consideration the setup for simulations which was carried out in this study, we must always bear in mind that the simulations highlight the path-dependent behaviour of CPPI 3, 5 and SLPI in contrast with the path-independency of the CPPI 1 and OBPI outcomes. For this reason, we separate the analysis making the comparison between the path-dependent strategies - CPPI 3 and 5 - and the path-independent strategies - CPPI 1 and OBPI. We must note that despite SLPI also being a path-dependent strategy, it is only so because of the two possible outcomes it can assume. Therefore, its distribution is very different than the distributions of the other path-dependent strategies. In this regard, we treat SLPI separately, because in some cases it can almost be path-independent, i.e., have one only possible outcome. Another consequence of simulating conditioned $S_T$, is that this study focuses only on high trend markets, because for negative returns, PI strategies return a value equal or insignificantly greater than the guarantee. In other words, taking the investors perspective, if we know that a stock will fall, we invest in a bond, a saving account, or simply do nothing. We are concerned to find in which cases cash-lock events occur for the CPPI 3 and 5, and which PI perform better under large positive market trends, i.e., assess to which extent these strategies really potentiate upside performance.

### 4.3.1 Path-Dependent Strategies and Cash-Lock

The first issue we address is that path-dependent strategies exhibit high cash-lock occurrences. For example, on a $40\%$ volatility market and maturity of 15 years, we can see that for every $S_T$ value, the payoffs of the path-dependent strategies end up cash-locked almost 100% of the simulations. This can be observed by the mean almost coinciding with the floor value, at the same time that the standard deviation ranges from values of the order of $10^{-3}$ to $10^{-2}$ for the CPPI 3, and from $10^{-6}$ to $10^{-10}$ for the CPPI 5. In these cases the low values of skewness and kurtosis for the CPPI 3 indicate us the non-existence of significant outliers and thus, almost no exceptions. Despite the high leptokurtic shape of CPPI 5, cash-lock events are even more frequent given the extremely low standard deviations. The reason for such frequency of cash-lock events is because a longer maturity is equivalent to a longer path which *ceteris paribus* amplifies path-dependency. But mostly, it is due to the high volatility, which increases the probability of larger drops in the underlying risky asset. Still looking at $\sigma = 40\%$, we see that even for a 5year-maturity investment, the path-dependent strategies do not escape a large set of cash-lock events. This can also be observed by the mean values - also near the floor - and standard deviations ranging from orders of $10^{-1}$ to 10 for the CPPI 3 and $10^{-3}$ to $10^{-1}$ for the CPPI 5. The only scenario where the path-dependent strategies perform better than the others, is for the combination of low volatility, short maturity and high returns of the risky asset: $S_T > 200$ with a guarantee floor of 100%, where the inequality loses its strictness for $\eta = 80\%$.

The SLPI is a rather peculiar strategy under the present framework's perspective. This strategy resumes to a two outcome lottery: either one receives the insured amount, or wins the risky asset as if it has been fully invested on it. The obtained probabilities of each outcome and a comparison with the other PI mean values, tell us that SLPI is probably the best choice in 6 cases, all of which with low volatility: for long maturity - 80% guarantee and $S_T \geq 200$ ; 100% guarantee and $S_T \geq 250$ - and for short maturity - 100% guarantee and $S_T = 200$. We carefully use the word probably because it is not clear for instance that an investor will prefer a SLPI which has 98.23%

probability of returning 300 with the remaining 1.77% chance of returning 100, as opposed to the OBPI strategy whose only possible outcome is $289.1\%$. This situation refers to the scenario of $T = 15$, $\sigma = 15\%$, $S_T = 300$ and $\eta = 100\%$.

### 4.3.2 Path-Independent Strategies

The study of the path-independent strategies is more direct in the present context. In general, the obtained moments show that the path-independent strategies are better suited for high volatile markets and longer maturities, regardless of the risky asset's payoffs. This is because they have less probability of being exaggeratedly invested on the risk-free asset. In particular, the CPPI 1 is better for moderate market increases and outperforms OBPI for a few cases of high volatility and long maturity. Conversely, the OBPI is a better choice than CPPI in some low volatile market scenarios.

### 4.3.3 Stochastic Dominance

So far we have identified in which scenarios path-dependent strategies are preferable than path-independent, and vice-versa on the perspective of the analysis of moments. However, in many cases, it is unclear only by the descriptive statistical analysis to grasp such conclusions. Therefore, we used stochastic dominance tests which take into account the whole cumulative distribution of the payoffs at maturity of two different strategies and provide an answer to whether an investor choses between those two strategies. Nevertheless, we see that the results of the stochastic dominant test confirm all the conclusions made with the analysis of moments. These results show in fact that investors who are simply interested in the higher payoffs, choose both CPPI 1 and OBPI over CPPI 3 and 5 strategies in almost all scenarios of high volatility. The same conclusions were also obtained for the dominance of the path-dependent strategies, which occurs only in low volatile markets, and short maturities. The SLPI exhibits dominance over CPPI 3 and 5 only on the combination of high volatility and long maturity. Between equally path-independent strategies, it becomes more clear with SD that in general CPPI 1 is chosen over OBPI for high volatile markets and longer maturities, while the opposite is observed for short maturity investments and low volatility. In addition, both dominate each other in different situations, CPPI 1 mainly for choices of $\eta = 100\%$ and OBPI for $\eta = 80\%$. As there have been many cases found of first order SD, few exceptions emerged for investors who can be both risk averse and decreasingly risk averse (second order SD), or for investors who have only the latter risk profile (third order). However, almost every second and third order of SD happen over SLPI.

## 5 Conclusion and Further Research

This study addresses an important issue concerning the path-dependency CPPI strategies which is extremely undesirable for investors and has not yet received an empirical study. This path dependency is directly related to the allocation mechanism of CPPIs and the fact that they often get cash-locked. This occurs because CPPIs tend to become excessively invested in the risk-free asset, which transgresses a fundamental purpose of PI: allow participation in upside performance of the risky asset. Hence the question that arises is: *When and how often do these cash-lock events happen?*, which leads necessarily to an even more important question: *Taking into consideration the* *cash-lock issue, which PI should an investor choose?*. In this work we provide an answer to both questions and emphasise the negative impact of this path-dependent behaviour on PI performance.

To answer the aforementioned questions, we begin by acknowledging that if we simulate risky asset paths all conditioned to the same final value, we obtain a single outcome for a path-independent strategy, while a path-dependent gives rise to a distribution. Hence, the difference between both types of strategies is highlighted with this approach, which is not encountered in previous studies on this subject. To achieve this, we assumed the risky asset follows a geometric Brownian motion which is a Gaussian process and can thus be simulated and conditioned to a fixed final value using Gaussian Processes for Machine Learning regression.

The main finding of this paper is that, in fact, cash-lock occurrences on the path-dependent CPPI 3 and 5 strategies happen very often and prohibit upside participation, even in cases where the risky asset triples at maturity. This is particularly patent on high volatile markets and for long maturities which is where the path-dependencies have more presence. Hence, under such market scenarios this undesirable risk makes the path-dependent strategies less attractive than the path-independent CPPI 1 and OBPI strategies. This conclusion is in consonance with previous studies and is corroborated with our analysis of the moments and stochastic dominance. However, the Buy and Hold strategy still remains a better choice for higher returns of the risky asset. Furthermore, in cases where volatility is low, the SLPI is almost identical to the Buy and Hold strategy. However, SLPI is more dependent on the risk profile of an investor and the stochastic dominance tests were not conclusive.

We conclude this paper with our goal achieved: to answer the questions posed above, presenting a different approach for the analysis of PI strategies. We also hope it contributes as a warning for investors who think of investing in CPPIs , which still need much improvement in the design process so that cash-lock risk is reduced.

We believe this topic alone has much more to be studied and discussed. In particular, there are other sources of path-dependency that can be introduced, e.g., borrowing constraints or different trading schedules. Such aspects can increase the cash-lock risk.

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